INTERNATIONAL CONFERENCE "COMPLEX AND HYPERCOMPLEX ANALYSIS AND THEIR APPLICATIONS"

UKRAINE, ZHYTOMYR
AUGUST 6 – 10, 2025

ABSTRACTS

Zhytomyr 2025

Institute of Mathematics of the National Academy of Sciences of Ukraine
Municipal Institution "Zhytomyr Regional Institute of Postgraduate Pedagogical Education" of Zhytomyr Regional Council
Hlybochytsia Village Council of Zhytomyr District of Zhytomyr Region

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The conference program includes two sections:

1. Complex and hypercomplex analysis and their application.

2. Problems of teaching methods of subjects and integrated courses of the physics and mathematics directions in the New Ukrainian School.

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ON THE SECOND DERIVATIVE AND ANGULAR ESTIMATES FOR A CLASS OF ANALYTIC FUNCTIONS

We establish a sharp upper bound for the the quantity $|g''(0)|$ for the function $g(z)$ in a class of analytic functions associated with the cardioid domain. Furthermore, a lower bound is obtained for the second angular derivative of the function $g(z)$ on the boundary of the unit disc, under an appropriate condition on $g'(1)$.

R E F E R E N C E S

1. G. M. Golusin, *Geometric Theory of Functions of Complex Variable*, Trans. of Mathematical Monographs 26, American Mathematical Society 1969.

2. I. S. Jack, Functions starlike and convex of order α. Journal of the London Mathematical Society **3** (1971), 469-474.

3. W. Janowski, *Some extremal problems for certain families of analytic functions I*, Annales Polonici Mathematici **28** (1973), 297-326.

4. W.C Ma and D. Minda, *A unified treatment of some special classes of univalent functions*, Proceeding of Conference on Complex Analytic, International Press, New York, (1994), 157-169.

5. R. Osserman, *A sharp Schwarz inequality on the boundary*, Proceedings of the American Mathematical Society **128** (2000), 3513-3517.

6. Ch. Pommerenke, *Boundary Behaviour of Conformal Maps*, Springer-Verlag, Berlin. 1992.

7. Kanika Sharma, Naveen Kumar Jain, V. Ravichandran, *Starlike functions associated with a cardioid,* Afr. Mat. **27** (2016), 923-939.

8. H. M. Srivastava, Gaganpreet Kaur and Gurmeet Singh, *Estimates of the fourth Hankel determinant for a class of analytic functions with bounded turnings involving cardioid domains*, Journal of Nonlinear and Convex Analysis, **22**(3) (2021), 511-526.

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SOME RESULTS FOR DRIVING POINT IMPEDANCE FUNCTIONS

In this study, the class of positive real functions has been considered as driving-point impedance functions which are frequently used in electrical engineering. Positive real functions are crucial particularly in both theoretical and applied circuit design. Here, we establish some sharp analytical inequalities related to the real part of Z(s). Considering these inequalities, the distribution and characteristics of the zeros and poles of the impedance function have been analyzed and corresponding generic circuit schematics are given with their spectral analyses.

For each of the derived inequalities, the extremal functions have been obtained via sharpness analysis. The extremal functions not only serve to show the sharpness of the inequalities but they also represent the properties of the corresponding circuit structures. The electrical circuits have been synthesized utilizing the obtained extremal functions and presented through schematic diagrams. Spectral analyses have also been carried out to examine how the theoretical results affect the frequency-domain behavior of the resulting circuits.

The results show that a variety of circuit structures can be obtained by applying considering the obtained extremal functions contributing to a deeper understanding of the relationship between the analytical properties of positive real functions and their physical circuit implementations. Consequently, the methodology presented in this study can possibly used for designing of passive electrical networks that meet specific performance criteria governed by mathematical bounds.

R E F E R E N C E S

1. Tavazoei, M. S. *Coefficient-based classes of algebraic conditions to construct positive real rational functions*, IEEE Trans. Circuits Syst. II, Exp. Briefs, **68**(7), 2374—2378 (2021)

2. Landau, E. and Valiron,G. *A deduction from Schwarz's lemma*, J. London Math. Soc., **1-4** (3), 162—163 (1929).

3. Nevanlinna, R. H. *Eindeutige Analytische Funktionen*, Springer, Berlin (1953).

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ON AN ATTEMPT TO INTRODUCE A NOTION OF BOUNDED INDEX FOR THE FUETER REGULAR FUNCTIONS OF THE QUATERNIONIC VARIABLE

Let $\mathbb{H}$ be the real associative algebra of quaternions with the standard basis 1, i, j, k such that $i^2 = j^2 = k^2 = -1$, $ij = -ji = k$, $jk = -kj = i$, $ki = -ik = j$. We write an element $q \in \mathbb{H}$ in the form $q = x_0 + ix_1 + jx_2 + kx_3$, where $x_\ell \in \mathbb{R}$, for $\ell = 0, 1, 2, 3$

Let $U \subseteq \mathbb{H}$ be an open set and let $f\colon U \to \mathbb{H}$ be a real differentiable function. We say that f is *left regular* on U

$$\frac{\partial_l f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + i\frac{\partial f}{\partial x_1} + j\frac{\partial f}{\partial x_2} + k\frac{\partial f}{\partial x_3} = 0.$$

We say that f is *right regular* on U if

$$\frac{\partial_r f}{\partial \bar{q}} = \frac{\partial f}{\partial x_0} + \frac{\partial f}{\partial x_1}i + \frac{\partial f}{\partial x_2}j + \frac{\partial f}{\partial x_3}k = 0.$$

The theory of the left regular functions is completely equivalent to the theory of the right regular functions so, classically, the theory is usually developed for the case of left regular functions.

Every regular function $f : \mathbb{H} \to \mathbb{H}$ can be represented as a uniformly convergent series (see [1,2])

$$f(q) = \sum_{n=0}^{\infty} \sum_{\nu \in \sigma_n} p_\nu(q - q_0) a_\nu,$$

where $a_\nu = \frac{(-1)^n}{n!}\partial_\nu f(q_0)$, $\partial_\nu = \frac{\partial^n}{\partial x_1^{n_1} \partial x_2^{n_2} \partial x_3^{n_3}}$, σ_n denotes the set of triples $\nu = (n_1, n_2, n_3)$, $n = n_1 + n_2 + n_3$, $q_0 \in \mathbb{H}$, and

$$p_\nu(q) = \sum_{1 \le \lambda_1, \ldots, \lambda_n \le 3} (x_0 \imath_{\lambda_1} - x_{\lambda_1}) \ldots (x_0 \imath_{\lambda_n} - x_{\lambda_n})$$

Here the sum is taken over the $\frac{n!}{n_1! n_2! n_3!}$ different alignments of n_i elements equal to i, with $i = 1, 2, 3$.

An entire function $f : \mathbb{C} \to \mathbb{C}$ is called a function of bounded index if there exists $n_0 \in \mathbb{Z}_+$ such that for all $z \in \mathbb{C}$ and for all $p \in \mathbb{Z}_+$

$$\frac{|f^{(p)}(z)|}{p!} \le \max\left\{\frac{|f^{(m)}(z)|}{m!} : 0 \le m \le n_0\right\}.$$

Below we introduce a generalization of the notion of index for the regular function of quaternionic variable.

A regular function $f : \mathbb{H} \to \mathbb{H}$ is said to be of bounded index if there exists $n_0 \in \mathbb{Z}_+$ such that for all $q \in \mathbb{H}$ and for all $\alpha = (\alpha_1, \alpha_2, \alpha_3) \in \mathbb{Z}_+^3$ the following inequality is true

$$\frac{|\partial_\alpha f(q)|}{(\alpha_1+\alpha_2+\alpha_3)!} \leq$$

$$\leq \max\left\{ \frac{|\partial_\beta f(q)|}{(\beta_1+\beta_2+\beta_3)!} :\ 0 \leq \beta_1+\beta_2+\beta_3 \leq n_0, \beta = (\beta_1, \beta_2, \beta_3) \in \mathbb{Z}_+^3 \right\}.$$

The least such integer n_0 is called the *index of the function* f and is denoted by $N(f, \mathbb{H})$.

It is possible to prove a similar proposition for the sum of two holomorphic functions, which is not a consequence of the previous one.

Theorem 1. ([3]) *Let $h_k : \mathbb{C} \to \mathbb{C}$ and their first order derivatives be entire functions of bounded index, $k \in \{1, 2, 3\}$. Then the function H : $bbH \to \mathbb{H}$ given by*

$$H(q) = \sum_{k=1}^{3} \operatorname{Re} h_k(x_0, x_k) + \sum_{k=1}^{3} \imath_k \, Im \, h_k(x_0, x_k)$$

is the Fueter-regular function of bounded index and its index does not exceed maximum of indexes of the functions h'_k increased by one.

Theorem 2. ([3]) *Let $h_1(z)$, $h_2(z)$ and their derivatives be entire functions of bounded index. Then the function $H(q)\colon \mathbb{H} \to \mathbb{H}$ defined as*

$$H(q) = h_1(x_0 + ix_1) + jh_2(x_2 + ix_3)$$

is the Fueter-regular function of bounded index and its index does not exceed maximum of indexes of the functions h'_k increased by one.

R E F E R E N C E S

1. F. Colombo, I. Sabadini, F. Sommen, D. C. Struppa, *Analysis of Dirac Systems and Computational Algebra*, Springer Science+Business Media LLC.

2. A. Sudbery, *Quaternionic analysis*, Mathematical Proceedings of the Cambridge Philosophical Society, **84**, 199B-225 (1979).

3. Lepson, B. *Differential equations of infinite order, hyperdirichlet series and entire functions of bounded index*, Proc. Sympos. Pure Math., **11**, 298–307 (1968).

4. Baksa, V.P., Bandura, A.I. *On an attempt to introduce a notion of bounded index for the Fueter regular functions of the quaternionic variable*, Matematychni Studii, **60**(2), 191-200 (2023).

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ON THE H-MEASURE OF AN EXCEPTIONAL SET IN FENTON-TYPE THEOREM FOR TAYLOR-DIRICHLET SERIES

We consider the class $S(\lambda, \beta, \tau)$ of convergent for all $x \geq 0$ Taylor-Dirichlet type series of the form

$$F(x) = \sum_{n=0}^{+\infty} a_n e^{x\lambda_n + \tau(x)\beta_n},$$

where $a_n \geq 0$ for all $n \geq 0$, $\tau : [0, +\infty) \to (0, +\infty)$ is continuously differentiable non-decreasing function, $\lambda = (\lambda_n)$ and $\beta = (\beta_n)$ are such that $\lambda_n \geq 0$, $\beta_n \geq 0$ for all $n \geq 0$.

At International conference "Complex Analysis and Related Topics"(Lviv, September 23-28, 2013) ([1]) the following conjecture was formulated.

Conjecture 1 ([1]). *The following statement is correct: For every sequences λ and β, functions τ, h, $\frac{h(x)}{x} \to +\infty$ $(x \to +\infty)$, there exist a function $F \in S(\lambda, \beta, \tau)$, a set E and a constant $d > 0$ such that $h - \mathrm{meas}\, E := \int\limits_E dh(x) = +\infty$ and $\forall x \in E$ the inequality $F(x) > (1 + d)\mu(x, F)$ holds.*

We give a partial answer to a question formulated in Conjecture 1.

Theorem 1 ([2]). *For each increasing function $h(x) : [0, +\infty) \to (0, +\infty)$, $h'(x) \nearrow +\infty$ $(x \to +\infty)$, every sequence $\lambda = (\lambda_n)$ such that*

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1} - \lambda_n} < +\infty$$

and for any non-decreasing sequence $\beta = (\beta_n)$ such that $\beta_{n+1} - \beta_n \leq \lambda_{n+1} - \lambda_n$ $(n \geq 0)$ there exist a function $\tau(x)$ such that $\tau'(x) \geq 1$ $(x \geq x_0)$, a function $F \in S(\lambda, \beta, \tau)$, a set E and a constant $d > 0$ such that h-meas $E := \int_E dh(x) = +\infty$ and

$$(\forall x \in E): \; F(x) > (1 + d)\mu(x, F),$$

where $\mu(x, F) = \max\{|a_n| e^{x\lambda_n + \tau(x)\beta_n} : n \geq 0\}$ is the maximal term of the series.

Given Theorem 1, the following questions arise.

Question 1. *Is the statement of Conjecture 1 correct in its entirety?*

Question 2 ([1]). *Let $h : \mathbb{R}_+ \to \mathbb{R}_+$ be a non-decreasing function such that $\frac{h(x)}{x} \to +\infty$, $(x \to +\infty)$. What are necessary and sufficient conditions that relationship*

$$F(x) = (1 + o(1))\mu(x, F)$$

holds for $x \to +\infty$ ($x \notin E$, h-meas$E < +\infty$) for every function $F \in S(\lambda, \beta, \tau)$?

The statement of Theorem 1 for the class $S(\lambda) := S(\lambda, 0, 0)$, that is, for the entire Dirichlet series, was proved earlier in paper [3].

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R E F E R E N C E S

1. Salo T.M., Skaskiv O.B., Trusevych O.M. *About exceptional sets in Fenton's type theorem*, Complex Analysis and Related Topics: Abstracts, International conf. (Lviv, September 23-28, 2013), Lviv, 2013, 67-68.

2. Bodnarchuk A., Gal' Yu., Skaskiv O. *On the h-measure of an exceptional set in Fenton-type theorem for Taylor-Dirichlet series*, Mat. Stud., **61** (2024), №1, 109–112. https://doi.org/10.30970/ms.61.1.109-112

3. Salo T.M., Skaskiv O.B., Trakalo O.M. *On the best possible description of exceptional set in Wiman-Valiron theory for entire funtions*, Mat. Stud. **16** (2001), №2, 131-140.

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ON PRODUCTS OF THE INNER RADII OF THE DOMAINS CONTAINING POINTS OF SOME STRAIGHT LINE

In the paper [1], the following problem was formulated.

Problem. Find the maximum value of the product

$$\prod_{k=1}^{m} r\left(B_k, a_k\right) \tag{1}$$

(where $r(B, a)$ is an inner radius of a domain B with respect to a point a) over various systems of non-overlapping domains $B_k \subset \overline{\mathbb{C}}$ and points $a_k \in B_k \cap [-1, 1]$, $k = 1, ..., m$, $m > 4$.

Later, this problem was repeated in the list of unsolved problems under number 17 in the Dubinin monograph. The following statement holds.

Theorem 1. [2] *Let $n \in \mathbb{N}$, $n \geqslant 2$; $l = \{z : z = z_0 + te^{i\varphi_0}\}$ is some straight line, where $z_0 \in \mathbb{C}$, $\varphi_0 \in \mathbb{R}$, are some fixed numbers, and $\varphi_0 \in (-\frac{\pi}{2}; \frac{\pi}{2})$, and t is a parameter that runs through the set of all real numbers. Then for any fixed points a_k, $k = \overline{1,n}$, line l, such that $a_k = z_0 + \rho_k e^{i\varphi_0}$, where ρ_k are real numbers, $0 = \rho_1 < \rho_2 < ... < \rho_n$, and any set of mutually non-overlapping domains $\{B_k\}$, $k = \overline{1,n}$, such that $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, the following inequality is valid:*

$$\prod_{k=1}^{n} r\left(B_k, a_k\right) \leqslant (n-1)^{-\frac{n}{4}} \left(\prod_{1 \leqslant p < k \leqslant n} |\rho_p - \rho_k| \right)^{\frac{2}{n-1}}.$$

The following theorem gives some estimate of the product (1) for the points a_k and domains B_k that satisfy all conditions of the above-formulated Problem.

Theorem 2. [2] *Let $n \in \mathbb{N}$, $n \geqslant 5$, $l = \{z : z = z_0 + te^{i\varphi_0}\}$ be some straight line, $a_1 = z_0$, $a_n = z_0 + \rho e^{i\varphi_0}$, where ρ is some positive real number. Then for an arbitrary set of points $\{a_k\}$, $k = \overline{2, n-1}$, in the interval (a_1, a_n) and an arbitrary set of domains $\{B_k\}$, $k = \overline{1,n}$, such that $a_k \in B_k \subset \overline{\mathbb{C}}$, $k = \overline{1,n}$, $B_i \cap B_j = \emptyset$, $i \neq j$, the following inequality holds:*

$$\prod_{k=1}^{n} r(B_k, a_k) \leqslant (n-1)^{-\frac{n}{4}} \prod_{k=1}^{n-1} \left(\frac{\rho k}{n-1} \right)^{\frac{2(n-k)}{n-1}}.$$

R E F E R E N C E S

1. Dubinin, V.N. *Capacities of condensers, generalizations of Grötzsch Lemmas, and symmetrization*, J. Math. Sci. **143**(3), 3053–3068 (2007).

2. Denega, I., Zabolotnyi, Y. *On products of the inner radii of the domains containing points of some straight line*, J. Math. Sci. **284**(3), 299-314 (2024).

3. Zabolotnyi, Y., Denega, I. *On products of inner radii of non-overlapping domains containing certain segment points*, Complex Variables and Elliptic Equations, https://doi.org/10.1080/17476933.2024.2416413

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RECONSTRUCTION OF AN ENTIRE FUNCTION IN A LOCALLY CONVEX SPACE FROM DISCRETE VALUES

The reconstruction of a signal from its discrete samples is a classical and fundamental problem in information theory. In modern applications, including communication systems, quantum computing, and image processing, there exists a need to study vector-valued signals modeled as functions taking values in locally convex spaces. These signals extend beyond purely numerical models and are used to represent abstract entities such as states, fields, or operators. However, generalizing classical reconstruction theory to infinite-dimensional settings introduces significant challenges, particularly those related to topological structures and convergence behavior. In this context, the application of Lagrange-type interpolation formulas entails several subtle and nontrivial analytical considerations.

Let us consider abstract entire analytic functions of a complex variable z $(z = x + iy)$, defined by a globally convergent power series of the form

$$f(z) = c_0 + c_1 z + c_2 z^2 + ... + c_n z^n + ...,$$

where the coefficients $c_i \in X$ and X is a locally convex space.

Definition. *A function $f(z)$ is called an entire function of exponential type σ (denoted $f(z) \in B_\sigma$), if for every $\epsilon > 0$ and for every seminorm $p_\alpha(.)$ there exists a constant $A_{(\epsilon,\alpha)} > 0$, such that for all z*

$$p_\alpha(f(z)) \le A_{(\epsilon,\alpha)} e^{(\sigma+\epsilon)|z|},$$

and for some sequence of points $\hat{z} \in \mathbb{C}$

$$p_\alpha(f(\hat{z}) > A_{(\epsilon,\alpha)} e^{(\sigma-\epsilon)|\hat{z}|}.$$

We denote by B_σ the set of abstract entire functions of exponential type σ that are bounded on the real axis in the sense that

$$p_\alpha(f(x)) \le M_\alpha, \quad x \in (-\infty, \infty).$$

Definition. *The symbol $W^{(2)}_{(*,\sigma)}$ denotes the class of abstract functions of exponential type σ, that are bounded on the real axis and possess weak square-integrability over the real line.*

$$f \in W^{(2)}_{(*,\sigma)} \Leftrightarrow \begin{cases} 1. \ f(x) \in B_\sigma \\ 2. \ \sup\limits_{-\infty<x<+\infty} |\langle x^*, f(x)\rangle| < \infty, \forall x^* \in X^* \\ 3. \ \int\limits_{-\infty}^{\infty} |\langle x^*, f(x)\rangle|^2 dx < \infty, \forall x^* \in X^* \end{cases}$$

where X^* is the dual space of the locally convex space X.

Remark. *In the case under consideration, it is straightforward to show that the following two conditions are equivalent:*

$$\int\limits_{\infty}^{\infty} |\langle x^*, f(x)\rangle|^2 dx < \infty \quad \text{and} \quad \sum_{\mathrm{n}=-\infty}^{\infty} |\langle \mathrm{x}^*, \mathrm{f}(\frac{\mathrm{n}\pi}{\beta})\rangle|^2 < \infty.$$

That is, $\forall x^* \in X^*$

$$\begin{cases} 1. \ f(x) \in B_\sigma \\ 2. \ \sup\limits_{-\infty<x<+\infty} |\langle x^*, f(x)\rangle| < \infty, \\ 3. \ \int\limits_{-\infty}^{\infty} |\langle x^*, f(x)\rangle|^2 dx < \infty, \end{cases} \Leftrightarrow \begin{cases} 1. \ f(x) \in B_\sigma \\ 2. \ \sup\limits_{-\infty<x<+\infty} |\langle x^*, f(x)\rangle| < \infty, \\ 3. \ \sum\limits_{n=-\infty}^{\infty} |\langle x^*, f(\frac{n\pi}{\beta})\rangle|^2 < \infty. \end{cases}$$

For abstract entire functions of exponential type σ, the following Lagrange interpolation formula can be stated:

Theorem 1. *Let $f(z) \in W^{(2)}_{(*,\pi)}$ be an abstract function taking values in a sequentially complete locally convex space X that contains no subspaces isomorphic to c_0. Then the function $f(z)$ can be uniquely reconstructed from the sequence $\{f(n)\}_{n=-\infty}^{\infty} \in X$ by the Lagrange interpolation formula*

$$f(z) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{\sin(\pi z)}{\pi(z-n)} f(n).$$

Theorem 2. *Let $f(z) \in B_\sigma$, $\sigma < \pi$ be an abstract function taking values in a sequentially complete locally convex space X that contains no subspaces isomorphic to c_0. Then the function $f(z)$ can be uniquely reconstructed from the sequence $\{f(n)\}_{n=-\infty}^{\infty} \in X$*

$$f(z+k) = \sum_{n=-\infty}^{\infty} (-1)^n \frac{\sin(\pi z)}{\pi(n-z)} \cdot \frac{\sin(\omega(n-z))}{\omega(n-z)} f(n+k), \quad \omega < \pi - \sigma.$$

R E F E R E N C E S

1. Akhiezer, N. I. *Theory of Approximation*, F. Ungar Publ. Comp., (1956).

2. Dimitrov, D. B. *On abstract functions taking values in a locally convex space not containing subspaces isomorphic to* c_0, Theory of Functions, Functional Analysis and Their Applications, **16**, 159-165 (1972).

3. Kandil, M. A. *On the Wiener-Paley theorem and some other problems in the theory of abstract entire functions*, Matematicheskii Sbornik,**73**(115), 3, 305-319 (1967).

4. Levin, B. Ya. *Distribution of zeros of entire functions*, AMS (1964)

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ON BOUNDARY DISTORTION ESTIMATES OF MAPPINGS IN DOMAINS WITH POINCARE INEQUALITY

Let $Q:\mathbb{R}^n\rightarrow [0,\infty]$ be a Lebesgue measurable function that vanishes outside D. In view of [section 7.6,1], we will say that the mapping $f:D\rightarrow \overline{\mathbb{R}^n}$ is a *ring Q-mapping at the point $x_0\in \overline{D}$ with respect to p-module,* $x_0\ne\infty$, $p\geqslant 1$, if there is $r_0=r(x_0)>0$ such that for arbitrary $0<r_1<r_2<r_0$ the following inequality holds

$$M_p(f(\Gamma(S(x_0,r_1),\,S(x_0,r_2),\,D)))\leqslant\int\limits_{A} Q(x)\cdot\eta^p(|x-x_0|)\,dm(x)\,, \qquad (1)$$

where $\eta:(r_1,r_2)\rightarrow [0,\infty]$ is an arbitrary non-negative Lebesgue-dimensional function such that

$$\int\limits_{r_1}^{r_2}\eta(r)\,dr\geqslant 1\,. \qquad (2)$$

Given [section 7.22,2], we say that the borel function $\rho\colon D\rightarrow [0,\infty]$ is *the upper gradient* for $u\colon D\rightarrow\mathbb{R}$, if the inequality $|u(x)-u(y)|\leqslant\int\limits_{\gamma}\rho\,|dx|$ is satisfied for all the smooth curves γ, connecting the points x and $y\in D$. We say that in the domain D the Poincare inequality $(1;p)$, $p\geqslant 1$, holds if there exist constants $C\geqslant 1$ and $\tau>0$ such that for each ball $B\subset D$, of an arbitrary bounded continuous function $u\colon D\rightarrow\mathbb{R}$ and each of its upper gradients ρ holds

$$\frac{1}{m(B)}\int\limits_{B}|u(x)-u_B|\,dm(x)\leqslant C\cdot(\operatorname{diam}B)\left(\frac{1}{m(\tau B)}\int\limits_{\tau B}\rho^p(x)\,dm(x)\right)^{1/p},$$

where $u_B := \frac{1}{m(B)}\int\limits_B u(x)\,dm(x)$. A domain D is called *Ahlfors regular,* if there exists a constant $C \geqslant 1$ such that for every $x_0 \in D$ and any $R < \operatorname{diam} D$ the inequalities $\frac{1}{C}R^n \leqslant m(B(x_0,R)\cap D) \leqslant CR^n$ holds. Let $A, B \subset \mathbb{R}^n$. Let's put $\operatorname{diam} A = \sup\limits_{x,y\in A}|x-y|$, $\operatorname{dist}(A,B) = \inf\limits_{x\in A, y\in B}|x-y|$.

For $\delta > 0$ and $p \geqslant 1$, the domains $D, D' \subset \mathbb{R}^n$, $n \geqslant 2$, $x_0 \in \partial D$, continuum $A \subset D$ and the Lebesgue measurable function $Q : D \to [0,\infty]$ denoted by $\mathfrak{F}^{p,x_0}_{Q,A,\delta}(D,D')$ the family of all ring Q-homeomorphisms f of D onto D' at the point x_0 with respect to the p-module, satisfying the condition $\operatorname{diam}(f(A)) \geqslant \delta$. The following result is obtained.

Theorem 1. *Let $x_0 \in \partial D$, $x_0 \neq \infty$, $n-1 < p \leqslant n$. Assume that D' is a regular Alfors bounded domain with $(1;p)$-Poincare inequality, and the following conditions are satisfied: 1) there exists $r_0' = r_0'(x_0) > 0$ such that the set $B(x_0,r)\cap D$ is connected for all $0 < r < r_0'$; 2) there exists $\delta_0 = \delta_0(x_0) > 0$ such that.*

$$\int\limits_{\varepsilon}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}<\infty \qquad \forall\quad \varepsilon\in(0,\delta_0)\,, \qquad \int\limits_{0}^{\delta_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}=\infty\,. \tag{3}$$

Then every $f \in \mathfrak{F}^{p,x_0}_{Q,A,\delta}(D,D')$ has a continuous extension to the point x_0, in addition, there are $\varepsilon_0 > 0$ and $\widetilde{C} > 0$, such that for all $x, y \in B(x_0,\varepsilon_0)\cap D$, $|x-x_0| \geqslant |y-x_0|$, and for all $f \in \mathfrak{F}^{p,x_0}_{Q,A,\delta}(D,D')$ is valid next evaluation

$$|f(x)-f(y)|\leqslant\widetilde{C}\cdot\left(\int\limits_{|x-x_0|}^{\varepsilon_0}\frac{dt}{t^{\frac{n-1}{p-1}}q_{x_0}^{\frac{1}{p-1}}(t)}\right)^{1-p}. \tag{4}$$

R E F E R E N C E S

1. *Martio O., Ryazanov V., Srebro U. and Yakubov E.* Moduli in Modern Mapping Theory. – New York: Springer Science + Business Media, LLC, 2009.

2. *Heinonen J.* Lectures on Analysis on metric spaces. New York: Springer Science+Business Media, 2001.

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AN ANALOGUE OF THE WIMAN THEOREM FOR ENTIRE FUNCTIONS OF SEVERAL VARIABLES

Let $\mathcal{L}$ be the class of positive continuous functions increasing to $+\infty$ on $[0;+\infty)$ and $\mathcal{L}_1$ be the class of continuous positive nondecreasing on $[0,+\infty)$ functions h such that $h(x+O(1)) = O(h(x))$ $(x \to +\infty)$.

Let $\mathcal{L}_2$ be the class of continuous positive nondecreasing on $[0;+\infty)$ functions h such that $h\big(x+1/h(x)\big) = O(h(x))$ $(x \to +\infty)$.

Let h be a positive continuous nondecreasing function and $E \subset [0;+\infty)$ be a locally Lebesgue measurable set of finite measure $\operatorname{meas} E = \int_E dx < +\infty$. Then the asymptotic h-density of E is defined as

$$D_h(E) = \overline{\lim_{R\to+\infty}}\, h(R)\cdot \operatorname{meas}(E \cap [R,+\infty)).$$

Let $A = (A_1,\ldots,A_p) \in \mathbb{R}^p$ be a fixed vector, and $\{G_{r,A}\}_{r\ge 0}$ be a system of A-like polylinear domains, which is the exhaustion of $\mathbb{C}^p$ ([1]).

Let us consider the class $\mathcal{H}^p$ of entire functions in $\mathbb{C}^p$, which are bounded in an arbitrary domain $\Pi_R = \{z \in \mathbb{C}^p\colon \ \mathrm{Re} z < R\}$, $R = (R_1,\ldots,R_p) \in \mathbb{R}^p_+$. For a function $F \in \mathcal{H}^p$ and $x \in \mathbb{R}^p$ it is obvious that $M(x,F) := \sup\{|F(x+iy)| : y \in \mathbb{R}^p\} < +\infty$. Moreover, if an analytic function F is bounded in a polylinear domain G, then for every x such that $\{z \in \mathbb{C}^p\colon \ \mathrm{Re}\, z_1 = x_1,\ldots,\mathrm{Re}\, z_p = x_p\} \subset G$ one has $M(x,F) < +\infty$.

For a function $F \in \mathcal{H}^p$ and $r > 0$ the supremum of its modulus at the polylinear domain $G_{r,A}$ is denoted by $S_F(r,A) := \sup\{|F(z)| : \ z \in G_{r,A}\}$. Since $\ln S_F(r,A)$ is a convex function, it has a right-hand derivative everywhere $L_F(r,A) := \big(\ln S_F(r,A)\big)'_+$,

For $F \in \mathcal{H}^p$ and $r > 0$ let us denote

$$B_F(r,A) = \sup\{\mathrm{Re}\ F(z)\colon z \in \partial G_{r,A}\}, \quad m_F(r,A) = \inf\{\mathrm{Re}\ F(z)\colon z \in \partial G_{r,A}\}.$$

The following theorem is an analog of Wiman's theorem.

Theorem 1 [1]. *Let $F \in \mathcal{H}^p$, $A \in \mathbb{R}^p$ be such that $L_F(r,A) \uparrow +\infty$ $(r \to +\infty)$. If $\Phi \in \mathcal{L}$, $h \in \mathcal{L}_2$ are the functions such that*

$$L_F(r,A) \ge \Phi(r)\, (r \ge r_0), \quad h(r) = o(\Phi(r)) \quad (r \to +\infty),$$

then there exists a set $E \subset \mathbb{R}_+$ of zero asymptotic h-density (i.e. $D_h(E) = 0$) such that

$$S_F(r,A) = (1+o(1))B_F(r,A) = -(1+o(1))m_F(r,A) \tag{1}$$

as $r \to +\infty$ $(r \in \mathbb{R}_+ \setminus E)$.

R E F E R E N C E S

1. Skaskiv O, Bandura A, Salo T, Dubei S. *Entire functions of several variables: Analogs of Wiman's theorem*, Axioms, **2025**, 14(3):216 (2025). https://doi.org/10.3390/axioms14030216

Nelson Faustino (Aveiro, Portugal)

DISPERSIVE PROPERTIES OF SPACE-FRACTIONAL DIRAC EQUATIONS AND RELATED MODELS

In this talk, we introduce a novel Cauchy problem driven by a fractional Dirac operator of the form $\boldsymbol{D}(-\Delta)^{\frac{\alpha-1}{2}}$ $(0<\alpha\leq 2)$, where the Dirac operator $\boldsymbol{D}$ is defined with respect to a Clifford algebra of signature $(0,n)$ and the fractional Riesz operator $(-\Delta)^{\frac{\alpha-1}{2}}$ is characterized by its Fourier symbol $|\boldsymbol{\xi}|^{\alpha-1}$. Accordingly, the Cauchy problem

$$\begin{cases} i\partial_t\boldsymbol{w}(\boldsymbol{x},t)+\boldsymbol{D}(-\Delta)^{\frac{\alpha-1}{2}}\boldsymbol{w}(\boldsymbol{x},t)=0, & (\boldsymbol{x},t)\in\mathbb{R}^{n+1}, \\ \boldsymbol{w}(\boldsymbol{x},0)=\boldsymbol{w}_0(\boldsymbol{x}), & \boldsymbol{x}\in\mathbb{R}^n. \end{cases} \tag{1}$$

serves as our starting point in unveiling the system's inherent dispersive properties. To this end, we employ Strichartz-type estimates for the unitary semigroup $\left\{e^{-it(-\Delta)^{\frac{\alpha}{2}}}\right\}_{t\in\mathbb{R}}$ in conjunction with Hardy space projectors $\frac{1}{2}(I\pm\mathcal{H})$, where $\mathcal{H}=\boldsymbol{D}(-\Delta)^{-\frac{1}{2}}$ denotes the Riesz-Hilbert transform. This significant theoretical insight was previously explored by the author in Reference 1. and has been extensively applied in Reference 2.

Furthermore, we establish a canonical correspondence between the solutions of the Cauchy problem (1) and those of the following nonlinear Cauchy problem

$$\begin{cases} \partial_t\boldsymbol{u}(\boldsymbol{x},t)-i(-\Delta)^{\frac{\alpha}{2}}\mathcal{H}\boldsymbol{u}(\boldsymbol{x},t)=\boldsymbol{D}\boldsymbol{G}(\beta\boldsymbol{u}(\boldsymbol{x},t)) & ,\ (\boldsymbol{x},t)\in\mathbb{R}^{n+1}, \\ \boldsymbol{u}(\boldsymbol{x},0)=\boldsymbol{u}_0(\boldsymbol{x}) & ,\ \boldsymbol{x}\in\mathbb{R}^n, \end{cases} \tag{2}$$

demonstrating that if $\boldsymbol{u}$ is a solution of the Cauchy problem (1) and there exists a Clifford-valued function $\boldsymbol{F}$ satisfying

$$\boldsymbol{D}\boldsymbol{F}(\boldsymbol{x},t)=\beta\boldsymbol{u}(\boldsymbol{x},t), \quad \text{for all} \quad (\boldsymbol{x},t)\in\mathbb{R}^{n+1},$$

then a suitable choice of the nonlinear function $\mathbf{G}$ guarantees that $\boldsymbol{w}=e^{-i\beta^{-1}\boldsymbol{F}}$ $(\beta\neq 0)$ is a solution of the nonlinear Cauchy problem (2).

The latter construction draws significant inspiration from Tao's seminal work on the Benjamin-Ono equation (see Reference 3.). Fundamentally rooted in gauge theory, it exhibits notable parallels with the Cole-Hopf transformation.

R E F E R E N C E S

1. Faustino, N. *On fundamental solutions of higher-order space-fractional Dirac equations*, Mathematical Methods in the Applied Sciences, **47**(10), 7988-8001 (2024).

2. Bernstein, S., Faustino, N. *Paley-Wiener type theorems associated to Dirac operators of Riesz-Feller type*, arXiv preprint arXiv:2405.04989 (2024).

3. Tao, T. *Global well-posedness of the Benjamin-Ono equation in* $H^1(\mathbb{R})$, Journal of Hyperbolic Differential Equations, **1**(01), 27-49 (2004).

Sergii Favorov (Kharkiv, Ukraine)

CRYSTALLINE MEASURES, fOURIER QUASICRYSTALS, THEIR GENERALIZATIONS AND SOME APPLICATIONS

We give definitions of crystalline measure and Fourier quasicrystal, discuss the relationship between these notions, and show some sufficient conditions for a crystalline measure to be a Fourier quasicrystal. We then present the theorem of Olevskii and Ulanovskii on the 1-1 correspondence between zeros of exponential polynomials and Fourier quasicrystals with unit masses. We show generalizations of this result to the zeros of absolutely convergence Dirichlet series, to slowly growing measures with countable spectrum, to the zeros of almost periodic functions in a strip. The last result yields the simple criterion of representability of almost periodic entire functions of exponential growth (in particular exponential polynomials) as a finite product of sines.

Cristina Flaut (Constanţa, România)

ABOUT OF k-POTENT ELEMENTS IN THE SPLIT QUATERNION ALGEBRA $\mathbb{H}_{\mathbb{Z}_p}$

In the following, we present the number of k-potent elements over $\mathbb{H}_{\mathbb{Z}_p}$ and we give a descriptive formula for the general case. For $k \in \{3, 4, 5\}$, we present an explicit formula for these values. The proposed methods give us all elements with these properties. Moreover, as an application, we count the number of solutions of the equation $x^k = 1$ over $\mathbb{H}_{\mathbb{Z}_p}$. For this purpose, we used computer as a tool to check and understand the behavior of these elements in each studied case. This presentation is based on the results obtained in the joint paper [FB; 25].

R E F E R E N C E S

1. Aristidou M., Demetre A., *A Note on Nilpotent Elements in Quaternion Rings over* $\mathbb{Z}_p$, International Journal of Algebra, 6(14)(2012), 663 - 666.

2. Fine N. J., Herstein I. N., *The Probability that a Matrix be Nilpotent*, Illinois Journal of Mathematics, 2(4A) (1958), 499-504.

3. Flaut C., Baias A., *Some Remarks Regarding Special Elements in Algebras Obtained by the Cayley–Dickson Process over* $\mathbb{Z}_p$, Axioms 3(6)(2024), 351; https://doi.org/10.3390/axioms13060351

4. Flaut C., Baias A., *The number of k-potent elements in the quaternion algebra* $\mathbb{H}_{\mathbb{Z}_P}$, https://arxiv.org/pdf/2412.02652

5. Flaut C, Hošková-Mayerová Š, Flaut D, *Models and Theories in Social Systems*, Springer Nature, 2019,
https://www.springer.com/us/book/9783030000837, e-ISBN: 978-3-030-00084-4, ISBN: 978-3-030-00083-7, DOI: 10.1007/978-3-030-00084-4

6. Miguel C. J., Serôdio R., *On the Structure of Quaternion Rings over* $\mathbb{Z}_p$, International Journal of Algebra, 5(27)(2011), 1313 - 1325.

7. Niven I., *The roots of a quaternion*, The American Mathematical Mounthly, 49(6)(1942), 386-388.

8. Schafer R. D., *An Introduction to Nonassociative Algebras,* Academic Press, New-York, 1966.

9. Voight J, *Quaternion Algebras*, Springer Nature Switzerland AG, ISBN 978-3-030-56692-0

10. Wongkumpra P., *Solutions to quadratic equations over finite fields*, Master of Science thesis, 2022,
https://digital_collect.lib.buu.ac.th/dcms/files/62910234.pdf

Anatoly Golberg (Holon, Israel)

WEAK CONFORMALITY AND EXTREMAL LENGTH BOUNDS

The notions of quasiconformality and quasiregularity in a domain are natural extensions of the conformality concept. The automorphism $f(z) = z(\sqrt{|z|}+1)/2$ of the unit disk provides a simple example of a quasiconformal mapping that preserves conformality only at the origin. The question of whether global quasiconformality/quasiregularity or their extensions can guarantee a mapping to be conformal at a prescribed point was raised more than 80 years ago, beginning with papers by Menshoff and Teichmüller. In our talk, we will present some local conditions weaker than conformality, illustrating them with several examples. All such results can be regarded as Teichmüller-Wittich-Belinskiĭ-type theorems.

Several approaches exist for studying the main features of quasiconformal and quasiregular mappings. The method of extremal lengths (moduli), which dates back to the classical works of Grötzsch and Ahlfors-Beurling, stands as one of the most fruitful techniques. An additional aim of our talk is to present new sharp bounds for the moduli of families of curves (paths), involving integrals containing the so-called directional dilatations. Several examples will illustrate these estimates.

Vladimir Gol'dshtein (Beer Sheva, Israel)
Evgeny Sevost'yanov (Zhytomyr, Ukraine)
Alexander Ukhlov (Beer Sheva, Israel)

ON THE THEORY OF GENERALIZED QUASICONFORMAL MAPPINGS

Let us give the basic definitions. Let Γ be a family of paths γ in $\mathbb{R}^n$. A Borel function $\rho : \mathbb{R}^n \to [0,\infty]$ is called *admissible* for Γ if $\int\limits_{\gamma} \rho(x)|dx| \geqslant 1$ for all (locally rectifiable) paths $\gamma \in \Gamma$. In this case, we write: $\rho \in \operatorname{adm}\Gamma$. Given a number $q \geqslant 1$, *q-modulus* of the family of paths Γ is defined as $M_q(\Gamma) = \inf\limits_{\rho\in \operatorname{adm}\Gamma} \int\limits_{D} \rho^q(x)\, dm(x)$. Let $x_0 \in \overline{D}$, $x_0 \neq \infty$, then

$$B(x_0, r) = \{x \in \mathbb{R}^n : |x - x_0| < r\}\,,$$

$$A = A(x_0, r_1, r_2) = \{x \in \mathbb{R}^n : r_1 < |x - x_0| < r_2\}\,.$$

Given sets E, $F \subset \overline{\mathbb{R}^n}$ and a domain $D \subset \mathbb{R}^n$, we denote $\Gamma(E, F, D)$ a family of all paths $\gamma : [a, b] \to \overline{\mathbb{R}^n}$ such that $\gamma(a) \in E, \gamma(b) \in F$ and $\gamma(t) \in D$ for all $t \in (a, b)$.

Let $Q : \mathbb{R}^n \to [0,\infty]$ be a Lebesgue measurable function. We say that f *satisfies the Poletsky inverse inequality with respect to q-modulus* at a point $y_0 \in f(D)$, $1 < q < \infty$, if the moduli inequality

$$M_q(\Gamma(E, F, D)) \leqslant \int\limits_{A(y_0,r_1,r_2)\cap f(D)} Q(y) \cdot \eta^q(|y - y_0|)\, dm(y) \qquad (1)$$

holds for any continua $E \subset f^{-1}(\overline{B(y_0, r_1)})$, $F \subset f^{-1}(f(D) \setminus B(y_0, r_2))$, $0 < r_1 < r_2 < r_0 = \sup\limits_{y\in f(D)} |y - y_0|$, and any Lebesgue measurable function $\eta : (r_1, r_2) \to [0,\infty]$ such that

$$\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1\,.$$

Let D, D' be domains in $\mathbb{R}^n$, $n \geqslant 2$. For numbers $1 \leqslant q < \infty$ and a Lebesgue measurable function $Q : \mathbb{R}^n \to [0, \infty]$, $Q = 0$ a.e. on $\mathbb{R}^n \setminus D'$, we denote be $\mathfrak{R}_Q^q(D, D')$ the family of all open and discrete mappings $f : D \to D'$ such that the moduli inequality (1) holds at any point $y_0 \in D'$.

Theorem. *Let $Q \in L^1(\mathbb{R}^n)$ and $q \geqslant n$. Suppose that, K is compact in D, and D' is bounded. Then there exists a constant $C = C(n, q, K, \|Q\|_1, D, D') > 0$ such that the inequality*

$$|f(x) - f(y)| \leqslant C_n \cdot \frac{(\|Q\|_1)^{\frac{1}{q}}}{\log^{\frac{1}{n}}\left(1 + \frac{r_0}{2|x-y|}\right)}\,, \ r_0 = d(K, \partial D),$$

holds for any $x, y \in K$ and $f \in \mathfrak{R}_Q(D, D')$, where $\|Q\|_1$ denotes the L^1-norm of the function Q in $\mathbb{R}^n$.

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PIECEWISE BIHARMONIC EXTENSION OF GRADIENTS AND MONOGENIC FUNCTIONS

Necessary and sufficient conditions have been found for the existence of an extension across a smooth curve for the gradients of functions defined and biharmonic in the corresponding domains adjacent to the given curve. Moreover, the found extension determines the gradient of the biharmonic function in the union of the indicated domains and has continuous partial derivatives of certain orders at the curve's points.

R E F E R E N C E S

1. Gryshchuk, S.V. *Piecewise biharmonic extension of gradients and monogenic functions*, J. Math. Sci., **288**(5), 590-603 (2025).

Gutlyanskii V. (Dnipro, Ukraine)
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Salimov R. (Kyiv, Ukraine)

TOWARD THE POTENTIAL THEORY IN ANISOTROPIC AND INHOMOGENEOUS MEDIA

The report is devoted to consequences from the well-developed theory of the Beltrami equations in the complex plane $\mathbb{C}$, see e.g. monographs [1]-[3], to the so-called generalized Cauchy-Riemann equations in the real plane $\mathbb{R}^2$, see [4],[5]. The former is a complex form of one of the main equations of hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media. The latter is a linear equation that relates, through a matrix-valued coefficient, the gradient of a stream function to the gradient of the potential function of steady state flow of incompressible fluid in such media. Our survey [5] includes, in particular, various types of results as theorems on the existence of regular solutions for the main boundary value problems of Hilbert, Dirichlet, Neumann, Poincare and Riemann, including non-linear, to the generalized Cauchy-Riemann equations.

As is well-known, the characteristic property of an analytic function $f = u + i\,v$ in the complex plane $\mathbb{C}$ is that its real and imaginary parts satisfy the **Cauchy–Riemann system**

$$\frac{\partial u}{\partial x} \;=\; \frac{\partial v}{\partial y}\,, \quad \frac{\partial u}{\partial y} \;=\; -\,\frac{\partial v}{\partial x}\,. \tag{1}$$

Euler was the first who found the connection of system (1) to analytic functions.

A physical interpretation of (1), going back to Riemann works on function theory, is that u and v represent a **potential function** and a **stream function** of the incompressible fluid steady flow in isotropic and homogeneous media.

The system (1) can be rewritten as the one equation in the matrix form

$$\nabla\, v \;=\; \mathbb{H}\, \nabla\, u\,, \tag{2}$$

where $\nabla\, v$ and $\nabla\, u$ denotes the gradient of v and u, correspondingly, interpreted as vector-columns in $\mathbb{R}^2$, and $\mathbb{H} : \mathbb{R}^2 \to \mathbb{R}^2$ is the so-called **Hodge operator** represented as the 2×2 matrix

$$\mathbb{H} \;:=\; \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}\,, \tag{3}$$

which carries out the counterclockwise rotation of vectors by the angle $\pi/2$ in the real plane $\mathbb{R}^2$. Thus, (2) shows that streamlines and equipotential lines of the fluid flow are mutually orthogonal.

Note that $\mathbb{H}$ is an analog of the imaginary unit i in the space $\mathbf{M^{2\times 2}}$ of all 2×2 matrices with real entries because

$$\mathbb{H}^2 \ := \ -I \ , \tag{4}$$

where I is the unit 2×2 matrix.

In the recent article [5], we have studied the interconnections between the Beltrami equations and the **generalized Cauchy-Riemann equations**

$$\nabla v \ = \ B\,\nabla u \tag{5}$$

with the matrix valued coefficients $B : D \to M^{2\times 2}$ that describe the incompressible fluid steady flows in anisotropic and inhomogeneous media. Then, on the basis of the well-developed theory of the Beltrami equations, we have given the corresponding consequences on the existence of regular solutions of the main boundary value problems for the equations (5).

Moreover, let us clarify the relationships of the equations (5) and the known $A-$**harmonic equation** for the potential function

$$\mathrm{div} A(Z)\,\mathrm{grad}\,u(Z) = 0 \ , \quad Z = (x,y) \in \mathbb{R}^2 \ , \tag{6}$$

with matrix valued coefficients $A : D \to M^{2\times 2}$ that is one of the main equations of hydromechanics (fluid mechanics) in anisotropic and inhomogeneous media.

For this purpose, recall that the Hodge operator $\mathbb{H}$ transforms curl-free fields into divergence-free fields and vice versa. Thus, if $u \in W^{1,1}_{\mathrm{loc}}$ is a solution of (6) in the sense of distributions, then the field $V := \mathbb{H}A\,\nabla u$ is curl-free and, consequently, $V = \nabla v$ for some $v \in W^{1,1}_{\mathrm{loc}}$ and the pair (u, v) is a solution of the equation (5) in the sense of distributions with

$$B \ := \ \mathbb{H}\cdot A \ . \tag{7}$$

Vice versa, if u and $v \in W^{1,1}_{\mathrm{loc}}$ satisfies (5) in the sense of distributions, then u satisfies (6) also in the sense of distributions with

$$A \ := \ -\mathbb{H}\cdot B \ = \ \mathbb{H}^{-1}\cdot B \tag{8}$$

because the curl of any gradient field is zero in the sense of distributions.

R E F E R E N C E S

1. Bojarski, B., Gutlyanskii, V., Martio, O., Ryazanov, V. *Infinitesimal Geometry of Quasiconformal and Bi-Lipschitz Mappings in the Plane*, EMS Tracts in Mathematics, 19. – European Mathematical Society (EMS): Zurich (2013) http://dx.doi.org/10.4171/122

2. Gutlyanskii, V., Ryazanov, V., Srebro, U., Yakubov, E. *The Beltrami Equation: Geometric Approach*, Developments in Mathematics, 26. - Springer: New York (2012) https://doi.org/10.1007/978-1-4614-3191-6

3. Martio, O., Ryazanov, V., Srebro, U., Yakubov, E. *Moduli in Modern Mapping Theory*, Springer Monographs in Mathematics - Springer: New York (2009) https://doi.org/10.1007/978-0-387-85588-2

4. Gutlyanski, V.Ya., Ryazanov, V.I., Salimov, A.R., Salimov, R.R. *On the Dirichlet problem for generalized Cauchy-Riemann equations*, Reports of the National Academy of Sciences of Ukraine, No 2. 3-10 (2025) https://doi.org/10.15407/dopovidi2025.02.003

5. Gutlyanski, V., Ryazanov, V., Salimov, A., Salimov, R. *On generalized Cauchy-Riemann equations for anisotropic and inhomogeneous media*, Journal of Mathematical Sciences, **288** (2), 199-224, (2025) https://doi.org/10.1007/s10958-025-07677-5

Oleh Herus (Zhytomyr, Ukraine)

ON THE ZYGMUND-TYPE ESTIMATE FOR A QUATERNIONIC SINGULAR CAUCHY INTEGRAL

1. Quaternionic singular Cauchy integral on a plane curve. Let $\mathbb{C} \supset \Gamma$ be a closed Jordan rectifiable curve, $\mathbb{H}(\mathbb{C})$ be the algebra of complex quaternions $a = \sum_{k=0}^{3} a_k \boldsymbol{i_k}$, where $\{a_k\}_{k=0}^{3} \subset \mathbb{C}$, $\boldsymbol{i}_0 = 1$, and $\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3$ be imaginary quaternionic units with the multiplication rule ${\boldsymbol{i}_1}^2 = {\boldsymbol{i}_2}^2 = {\boldsymbol{i}_3}^2 = \boldsymbol{i}_1\boldsymbol{i}_2\boldsymbol{i}_3 = -1$, $f : \Gamma \to \mathbb{H}(\mathbb{C})$ be a continuous function. We consider the *quaternionic singular Cauchy integral*

$$F_\alpha[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_\alpha(\zeta - t)\,\sigma\,(f(\zeta) - f(t)), \qquad t \in \Gamma,$$

where $\alpha \in \mathbb{C}$ and $t := \tau\boldsymbol{i}_1 + \eta\boldsymbol{i}_2$, $\zeta := \xi\boldsymbol{i}_1 + \eta\boldsymbol{i}_2$ are real quaternions from Euclidean space $\mathbb{R}^2$ equipped with the quaternion structure and $\sigma := d\eta\boldsymbol{i}_1 - d\xi\boldsymbol{i}_2$ is the quaternionic differential form, $\Gamma_{t,\delta} := \{\zeta\in\Gamma : |\zeta - t| \leqslant \delta\}$,

$$K_\alpha(z) := -\frac{z}{2\pi|z|^2} + \frac{\alpha}{2\pi}\ln|z| + \widetilde{K}_\alpha(z)$$

($\widetilde{K}_\alpha$ is a continuous function, vanishing when $\alpha = 0$) is the *quaternionic Cauchy kernel.*

Let for $\delta > 0$

$$\omega_\Gamma(f,\delta) := \sup_{\substack{|z_1-z_2|\leqslant\delta \\ \{z_1;z_2\}\subset\Gamma}} |f(z_1) - f(z_2)|$$

be the modulus of continuity of a function f on Γ. We obtain an upper estimate for the modulus of continuity of the integral $F_\alpha[f]$ in terms of the modulus of continuity of the integrand f and a metric characteristic of the curve Γ.

2. Quaternionic singular Cauchy integral on a space surface. For a closed Jordan rectifiable oriented surface $\Gamma \subset \mathbb{R}^3$ and for a continuous function $f : \Gamma \to \mathbb{H}(\mathbb{C})$ we consider the *quaternionic singular Cauchy integral*

$$F_\alpha[f](t) := \lim_{\delta\to 0} \int\limits_{\Gamma\setminus\Gamma_{t,\delta}} K_\alpha(\zeta - t)\,\nu(\zeta)\,(f(\zeta) - f(t))\,ds_\zeta, \qquad t \in \Gamma,$$

where $\alpha \in \mathbb{C}$, ds_ζ is the surface square element and $\nu(\zeta) := \nu_1(\zeta)\boldsymbol{i}_1 + \nu_2(\zeta)\boldsymbol{i}_2 + \nu_3(\zeta)\boldsymbol{i}_3$ is the unit outward normal vector to oriented surface Γ in those points $\zeta \in \Gamma$, where it exists,

$$K_\alpha(z) := -\left(\alpha + \frac{z}{|z|^2} + i\alpha\frac{z}{|z|}\right)\frac{e^{-i\alpha|z|}}{4\pi|z|}$$

is the *quaternionic Cauchy kernel.*

Let $\theta_z(\delta) := \operatorname{mes}\Gamma_{z,\delta}$ be the surface measure of the set $\Gamma_{z,\delta}$. The surface Γ is called a *regular surface*, if there exists a positive constant K, such as for all $z \in \Gamma$ and for all $\delta > 0$ the inequality $\theta_z(\delta) \leqslant K\delta^2$ holds true. We proved the next estimate for modulus of continuity of the integral $F_\alpha[f]$ on a regular surface Γ in terms of the modulus of continuity of the integrand f:

$$\omega_\Gamma(F_\alpha,\delta) \leqslant cKe^{3|\alpha|d}\left((1+2d|\alpha|)\int\limits_0^{2d}\frac{\omega_\Gamma(f,x)}{x\left(1+\dfrac{x}{\delta}\right)}dx + |\alpha|^2\delta\int\limits_\delta^{2d}\omega_\Gamma(f,x)\,dx\right),$$

where d is the diameter of the surface Γ.

REFERENCES

1. Moisil G. C., Theodoresco N. *Functions holomorphes dans l'espace*, Mathematica (Cluj) **5**, 142–159 (1931).

2. Sudbery A. *Quaternionic Analysis*, Math. Proc. Camb. Phil. Soc. **85**, 199–225 (1979).

3. Мейлихзон А. С. *По поводу моногенности кватернионов*, ДАН СССР **59**(3), 431–434 (1948).

4. Blaya R. A., Reyes J. B. *Boundary value problems for quaternionic monogenic functions on nonsmooth surfaces,* Advances in Applied Clifford Algebras **9**(1), 1–22 (1999).

5. Kravchenko V. V., Shapiro M. V. *Integral representations for spatial models of mathematical physics,* Addison Wesley Longman, Pitman Research Notes in Mathematics Series **351** (1996).

6. Vladimirov V. S. *Equations of mathematical physics,* New York: Marcel Dekker, Inc. (1971).

7. Blaya R. A., Reyes J. B., Shapiro M. V. *On the Laplasian vector fields theory in domains with rectifiable boundary,* Mathematical Methods in the Applied Sciences **29** (2006), 1861–1881.

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BANACH ALGEBRAS OF CONVOLUTION TYPE OPERATORS WITH OSCILLATING DATA ON WEIGHTED LEBESGUE SPACES

Let $\mathcal{B}_{p,w}$ be the Banach algebra of all bounded linear operators acting on the weighted Lebesgue space $L^p(\mathbb{R}, w)$, where $p \in (1, \infty)$ and w is a Muckenhoupt weight. Consider the Banach algebra $\mathfrak{A}_{p,w} \subset \mathcal{B}_{p,w}$ generated by all multiplication operators aI ($a \in PQC$) and all convolution operators $W^0(b)$ ($b \in PSO^{\diamond}_{p,w}$), where $PQC \subset L^\infty(\mathbb{R})$ is the C^*-algebra of piecewise quasicontinuous functions, $PSO^{\diamond}_{p,w} \subset M_{p,w}$ is the Banach algebra of piecewise slowly oscillating functions that admit piecewise slowly oscillating discontinuities at arbitrary points of $\mathbb{R} \cup \{\infty\}$, and $M_{p,w}$ is the Banach algebra of Fourier multipliers on $L^p(\mathbb{R}, w)$. For any $p \in (1, \infty)$ and any Muckenhoupt weight w, we study the Fredholmness of operators in the Banach algebra $\mathcal{Z}_{p,w} \subset \mathfrak{A}_{p,w}$ generated by the operators $aW^0(b)$ with quasicontinuous functions $a \in QC$ and slowly oscillating functions $b \in SO^{\diamond}_{p,w}$. This allows one to investigate the Fredholmness of operators $A \in \mathfrak{A}_{p,w}$ by using the Allan-Douglas local principle and the two idempotents theorem.

Vladimir V. Kisil (Leeds, England)

TRANSMUTATIONS FROM THE COVARIANT TRANSFORM ON THE HEISENBERG GROUP AND AN EXTENDED UMBRAL PRINCIPLE

We discuss several seemingly assorted objects: the umbral calculus [1], generalised translations [2] and associated transmutations [3-5], symbolic calculus of operators [6]. The common framework for them is representations of the Weyl algebra of the Heisenberg group by ladder operators [7]. Transporting various properties between different implementations we review some classic results and new opportunities [8].

REFERENCES

1. G.-C. Rota. *Finite Operator Calculus.* Academic Press, Inc., New York, 1975.

2. B. M. Levitan. *Teoriya Operatorov Obobwennogo Sdviga.* Izdat. "Nauka", Moscow, 1973.

3. I. A. Kipriyanov. *Singulyarnye Elliptiqeskie Kraevye Zadachi.* Fizmatlit "Nauka", Moscow, 1997. ISBN 5-02-014799-0.

4. S. Sitnik and E. Shishkina. *Method of Transmutations for Differential Equations with Bessel Operators.* Fizmatlit, Moscow, 2019.

5. V. V. Kravchenko and S. M. Sitnik, eds., *Transmutation Operators and Applications*, Springer, Cham, 2020.

6. V. V. Kisil. *Calculus of operators: Covariant transform and relative convolutions.* Banach J. Math. Anal., **8**(2), 156-184, 2014. `arXiv:1304.2792`,

7. R. Howe. *On the role of the Heisenberg group in harmonic analysis.* Bull. Amer. Math. Soc. (N.S.), **3**(2), 821-843, 1980.

8. V. V. Kisil, *Transmutations from the covariant transform on the Heisenberg group and an extended umbral principle*, Lobachevskii J. Math., **44**(8), 3384-3397, 2023. `arXiv:2305.14227`.

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GENERALIZATION OF FARADAY PROBLEM FOR THE MECHANICAL SYSTEM "RESERVOIR – LIQUID WITH A FREE SURFACE"

Structures with free surface liquid are often used in modern engineering as components of transportation and energy systems, as well as systems for the transportation and storage of various environmentally hazardous and technological liquids. In this case, the relative mass of the liquid can be significant, and thus the mobility of the liquid significantly affects the dynamics and control of such systems. Mathematical models of the dynamics of liquid-containing structures belong to a subclass of objects with internal degrees of freedom, and in cases of large relative mass of the liquid, it is crucial to account for the coupled nature of the motion of the system's components, which is traditionally a difficult task.

The parametric resonance in the mechanical system "tank -– liquid with a free surface" was first experimentally investigated by Faraday in 1831 [1]. A cylindrical tank, partially filled with water, was set up on special laboratory equipment and had the ability to move in a vertical plane according to a specified harmonic law. The result of the experiment was Faraday's conclusion of the fact that the first resonance frequency of the free surface of the liquid is equal to half of the frequency of the tank's disturbance. This result is known in the history of mechanics as the classical Faraday problem.

Since in the classical Faraday problem the reservoir moves only vertically according to a given law, the oscillations of the liquid do not affect the nature of its motion. In fact, this means that the reservoir moves in a vertical channel or has an infinitely large mass. However, in most practical applications (such as a ship pitching on waves, the flight of a launch vehicle, etc.), the structure with the liquid can undergo translational and angular motions in different planes due to both oscillations of the free surface of the liquid and the presence of external force or torque disturbances. In this case, the mass of the liquid can significantly exceed the mass of the tank, which is why considering the combined motion of the tank and the liquid with a free surface, as well as their interaction, is a determining factor. Thus, to approach practical applications, it is suggested to generalize the classical Faraday problem by introducing an additional degree of freedom to the system "tank — liquid with a free surface": the possibility of translational motion in the horizontal plane and the possibility of performing angular oscillations on a pendulum suspension.

A cylindrical tank, partially filled with liquid, is considered. We treat the tank as a rigid body that can undergo translational movement and angular oscillations due to active external forces and moments, as well as the presence of kinematic disturbances. The liquid is assumed to be ideal, incompressible, and homogeneous, and its initial motion is non-vortical. The work applies a mathematical model created in the works of O.S. Lymarchenko [2], which has undergone extensive validation, including comparisons with the results of theoretical and experimental research by other authors. The independent parameters chosen are the excitation amplitudes of the waveforms at the free surface of the liquid, and the translational and rotational motion parameters of the carrier structure. According to the theorem stating that the irrotational motion of an ideal incompressible homogeneous fluid is entirely determined by the motion of its boundaries (the amplitudes of the oscillation shapes determine the motion of the free surface of the fluid, and the linear and angular displacements determine the motion of the rigid boundaries of the area occupied by the fluid), the chosen parameters fully characterize the dynamics of the system and constitute a minimal set of parameters. At the same time, based on these parameters, the characteristics of the motion of the free surface of the fluid, the velocity field of the fluid, and the angular velocity of the motion of the tank can be completely restored.

To summarize Faraday problem, the following has been done: 1) a mechanical formulation of the problem for each case of generalization; 2) obtaining in analytical form formulas for calculating the natural frequencies of coupled vibrations; 3) obtaining in analytical form formulas for calculating the regions of stability and instability of the system; 4) conducting qualitative and spectral analysis of the results of computational experiments to identify the properties of the system's motion and to verify the validity of the accepted hypotheses.

The introduction of an additional degree of freedom into the system (the possibility of horizontal movement or angular oscillations of the tank on a pendulum suspension) leads to an increase in the frequency of parametric resonance, which is higher the smaller the mass of the tank relative to the mass of the liquid or the shorter the length of the pendulum suspension. When the tank is subjected to vertical perturbation, allowing for horizontal movement or angular oscillations on a pendulum suspension, unlike the classical Faraday problem, the dynamic processes in the system develop as a combination of parametric resonance and forced oscillations. For this generalization of the Faraday problem, the system can enter a nonlinear oscillation mode at any frequency. Generally, achieving a steady oscillation regime in nonlinear multi-frequency systems of the "tank – liquid with a free surface" type under parametric disturbances of the tank's movement does not occur – the spectrum of oscillations of the disturbance at the free surface of the liquid at the wall (as the most characteristic point available for measurements and observations) always contains harmonics at both the natural and combination frequencies.

R E F E R E N C E S

1. Ibrahim, R.A. *Liquid sloshing dynamics: theory and applications*, Cambridge University Press, Cambridge (2005).

2. Limarchenko, O.S. *Nonlinear dynamics of structures with liquid*, KPI, Kyiv (1997).

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ON ONE ANALOG OF NÄKKI THEOREM FOR NON-CONFORMAL MODULI

Let us give some definitions. A Borel function $\rho:\mathbb{R}^n\rightarrow[0,\infty]$ is called *admissible* for family Γ of paths γ in $\mathbb{R}^n$, if $\int\limits_{\gamma}\rho(x)\,|dx|\geqslant 1$ holds for any (locally rectifiable) path $\gamma\in\Gamma$. In this case, we write: $\rho\in\operatorname{adm}\Gamma$. Given $p\geqslant 1$, we define *p-modulus* of the family Γ by the equality $M_p(\Gamma)=\inf\limits_{\rho\in\operatorname{adm}\Gamma}\int\limits_{\mathbb{R}^n}\rho^p(x)\,dm(x)$. Set $M(\Gamma):=M_n(\Gamma)$. Let $E_0,E_1\subset\overline{\mathbb{R}^n}$, $\overline{\mathbb{R}^n}=\mathbb{R}^n\cup\{\infty\}$, and let D be a domain in $\overline{\mathbb{R}^n}$, $n\geqslant 2$. Set $M(\Gamma)=M_n(\Gamma)$. Denote by $\Gamma(E_0,E_1,D)$ the family of all paths joining E_0 and E_1 in D. The following result was established by R. Näkki (see [1, Lemma 1.15]).

Theorem A. (The positivity of the modulus of families of paths joining a pair of continua). *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$. If A and A^* are (nondegenerate) continua in D, then $M(\Gamma(A,A^*,D))>0$.*

We have proved Näkki's result to the case of a modulus of order $p\geqslant 1$. The following theorem holds.

Theorem A_1. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, and let $p>n-1$. If A and A^* are (nondegenerate) continua in D, then $M_p(\Gamma(A,A^*,D))>0$.*

Let $p\geqslant 1$ and let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$. We say that ∂D is *strongly accessible at the point $x_0\in\partial D$ with respect to p-modulus,* if for any neighborhood U of x_0 there is a compact set $E\subset D$, a neighborhood $V\subset U$ of x_0 and a number $\delta>0$ such that $M_p(\Gamma(E,F,D))\geqslant\delta$ for each continuum F in D with $F\cap\partial U\neq\varnothing\neq F\cap\partial V$.

Theorem B. *Let D be a domain in $\mathbb{R}^n$, $n\geqslant 2$, which has strongly accessible boundary at the point $x_0\in\partial D\setminus\{\infty\}$ with respect to p-modulus, $p>n-1$. Then D is finitely connected at the point x_0, in other words, for any neighborhood U of x_0 there exists a neighborhood $V\subset U$ of the same point such that $V\cap D$ has a finite number of components.*

R E F E R E N C E S

1. Näkki, R. *Boundary behavior of quasiconformal mappings in n-space*, Ann. Acad. Sci. Fenn. Ser. A. **484**, 1–50 (1970).

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HAUSDORFF ANALYTIC FUNCTIONS IN A THREE-DIMENSIONAL NONCOMMUTATIVE ALGEBRA

Let apply the Hausdorff approach [1] to the functions with values in the algebra $\widetilde{\mathbb{A}}_2$ over the field of complex numbers $\mathbb{C}$ with the basis $\{I_1, I_2, \rho\}$, whose elements satisfy the following multiplication rules:

$\cdot$	I_1	I_2	ρ
I_1	I_1	0	0
I_2	0	I_2	ρ
ρ	ρ	0	0

.

Let $\Omega = \{\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3 \rho \in \widetilde{\mathbb{A}}_2\}$ be a domain in the algebra $\widetilde{\mathbb{A}}_2$. Further, we will identify a domain Ω with the domain $\Omega_{\mathbb{C}} = \{(\xi_1, \xi_2, \xi_3) \in \mathbb{C}^3\}$.

A function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ of the form

$$\Phi(\zeta) = f_1(\xi_1, \xi_2, \xi_3) I_1 + f_2(\xi_1, \xi_2, \xi_3) I_2 + f_3(\xi_1, \xi_2, \xi_3)\rho, \tag{1}$$

is called *H-analytic* in a domain Ω, if Φ is differentiable by Hausdorff at every point $\zeta \in \Omega$, i.e., if the differential $d\Phi$ is a $\mathbb{C}$-linear homogeneous polynomials of the differential $d\zeta = d\xi_1 I_1 + d\xi_2 I_2 + d\xi_3 \rho$:

$$d\Phi = \sum_{s=1}^{9} A_s(\zeta)\, d\zeta\, B_s(\zeta)\,, \tag{2}$$

where A_s and B_s are some $\widetilde{\mathbb{A}}_2$-valued functions of the variable ζ.

In this case, for any $\zeta \in \Omega$ the element

$$\Phi'_H(\zeta) := \sum_{s=1}^{9} A_s\, B_s$$

is called *H-derivative* of the function Φ at the point ζ.

Theorem 1 [2]. *A function $\Phi : \Omega \to \widetilde{\mathbb{A}}_2$ of the form* (1)*, where $f_k : \Omega \to \mathbb{C}^3$ are holomorphic functions, is H-analytic in the domain $\Omega \subset \widetilde{\mathbb{A}}_2$ if and only if the conditions*

$$\frac{\partial f_1}{\partial \xi_2} = \frac{\partial f_1}{\partial \xi_3} = \frac{\partial f_2}{\partial \xi_1} = \frac{\partial f_2}{\partial \xi_3} = 0, \tag{3}$$

$$\frac{\partial^2 f_3}{\partial \xi_3^2} = 0 \tag{4}$$

are satisfied.

Theorem 2 [2]. *If the function $\Phi : \Omega \to \tilde{\mathbb{A}}_2$ is H-analytic in the domain Ω, then its derivative $\Phi'_H(\zeta)$ exists and is independent of a choice of the functions A_s, B_s in equality* (2). *In this case,*

$$\Phi'_H(\zeta) = \frac{\partial f_1}{\partial \xi_1} I_1 + \frac{\partial f_2}{\partial \xi_2} I_2 + \left(\frac{\partial f_3}{\partial \xi_1} + \frac{\partial f_3}{\partial \xi_2} \right) \rho.$$

Due to equalities (3) and taking into account that the functions f_1, f_2 are holomorphic, we have

$$f_1(\xi_1, \xi_2, \xi_3) = f_1(\xi_1), \quad f_2(\xi_1, \xi_2, \xi_3) = f_2(\xi_2).$$

Now integrating the equality (4) twice, we have

$$\frac{\partial f_3}{\partial \xi_3} = \varphi_1(\xi_1, \xi_2),$$

$$f_3(\xi_1, \xi_2, \xi_3) = \varphi_1(\xi_1, \xi_2)\xi_3 + \varphi_2(\xi_1, \xi_2),$$

where φ_1, φ_2 are holomorphic functions of two complex variables.

Therefore, every H-analytic function can be expressed in the form

$$\Phi(\zeta) = f_1(\xi_1)I_1 + f_2(\xi_2)I_2 + [\varphi_1(\xi_1, \xi_2)\xi_3 + \varphi_2(\xi_1, \xi_2)]\, \rho.$$

It is easy to show that the inverse element to the element $\zeta = \xi_1 I_1 + \xi_2 I_2 + \xi_3 \rho$ is of the form

$$\zeta^{-1} = \frac{1}{\xi_1}\, I_1 + \frac{1}{\xi_2}\, I_2 - \frac{\xi_3}{\xi_1 \xi_2}\, \rho, \tag{5}$$

where $\xi_1 \neq 0$ and $\xi_2 \neq 0$.

Let $\tau := t_1 I_1 + t_2 I_2$, where $t_1, t_2 \in \mathbb{C}$ and $d\tau := dt_1 dt_2$. Then, the next equality follows from equality (5):

$$(\tau - \zeta)^{-1} = \frac{1}{t_1 - \xi_1}\, I_1 + \frac{1}{t_2 - \xi_2}\, I_2 - \frac{\xi_3}{(t_1 - \xi_1)(t_2 - \xi_2)}\, \rho.$$

Let $D \subset \mathbb{C}^2$ be an arbitrary domain on the plane $(t_1,\, t_2)$. In the domain D we take an arbitrary subdomain G such that $G = G_1 \times G_2$, G_s is a domain on a plane t_s, $s = 1, 2$, with a piecewise smooth boundary ∂G_s, and $\xi_s \in G_s$, $s = 1, 2$. In follow $\Gamma := \partial G_1 \times \partial G_2$.

Now we construct a representation of H-analytic functions as the sum of integrals of the form

$$\frac{1}{(2\pi i)^2} \int_{\Gamma} f(t_1, t_2)\, (\tau - \zeta)^{-1} d\tau,$$

where $f(t_1,t_2)$ is some holomorphic function of two complex variables in D.

Theorem 3 [2]. *Every H-analytic function $\Phi : D\times\mathbb{C} \to \widetilde{\mathbb{A}}_2$ in a domain $D\times\mathbb{C}$ can be expressed by formula*

$$\Phi(\zeta) = \frac{1}{(2\pi i)^2}\int\limits_\Gamma \left(\frac{f_1(t_1)}{t_2-\xi_2}\, I_1 - \varphi_1(t_1,t_2) I_2 + \frac{\varphi_2(t_1,t_2)}{t_2-\xi_2}\,\rho\right)(\tau-\zeta)^{-1}d\tau +$$

$$+\frac{1}{(2\pi i)^2}\int\limits_\Gamma \frac{f_2(t_2)}{t_1-\xi_1}(\tau-\zeta)^{-1}d\tau\, I_2\,,$$

where f_1, f_2 are some holomorphic functions in the domain D.

R E F E R E N C E S

1. Hausdorff F. *Zur Theorie der Systeme complexer Zahlen*, Leipziger Berichte, **52**, 43–61 (1900).

2. Shpakivskyi V. S., Kuzmenko T. S. *Hausdorff analytic functions in the three-dimensional associative noncommutative algebra*, J. Math. Sci. **262**, 207–221 (2022).

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THE RIGHT–LEFT WG QUATERNION INVERSES

Let $\mathbb{H}^{m\times n}$ be the set of all $m\times n$ matrices over the quaternion skew field $\mathbb{H}$.

If $\mathbf{A}\in\mathbb{H}^{n\times m}$, its *Moore-Penrose inverse* is unique $\mathbf{X}=\mathbf{A}^\dagger$ such that

$$(1)\ \mathbf{AXA}=\mathbf{A},\ (2)\ \mathbf{XAX}=\mathbf{X},\ (3)\ (\mathbf{AX})^*=\mathbf{AX},\ (4)\ (\mathbf{XA})^*=\mathbf{XA}.$$

The *Drazin inverse* $\mathbf{A}^D$ of $\mathbf{A}\in\mathbb{H}^{n\times n}$ with $k=\mathrm{Ind}(\mathbf{A})=\min\{k\in\mathbb{N}\cup\{0\}\mid \mathrm{rank}(\mathbf{A}^k)=\mathrm{rank}(\mathbf{A}^{k+1})\}$ is unique $\mathbf{X}$ for which

$$(1^k)\ \mathbf{A}^k\mathbf{XA}=\mathbf{A}^k,\ (2)\ \mathbf{XAX}=\mathbf{X},\ (5)\ \mathbf{AX}=\mathbf{XA}.$$

The *right core–EP inverse* $\mathbf{A}^{\textcircled{\dagger}}$ of $\mathbf{A}\in\mathbb{H}^{n\times n}$ with $k=\mathrm{Ind}(\mathbf{A})$ presents the distinctive solution to Eqs. $\mathbf{X}=\mathbf{XAX}$ and $\mathcal{C}_r(\mathbf{X})=\mathcal{C}_r(\mathbf{A}^D)=\mathcal{C}_r(\mathbf{X}^*)$. The *left core–EP inverse* $\mathbf{A}_{\textcircled{\dagger}}$ of $\mathbf{A}\in\mathbb{H}^{n\times n}$ is the unique solution to $\mathbf{X}=\mathbf{XAX}$ and $\mathcal{R}_l(\mathbf{X})=\mathcal{R}_l(\mathbf{A}^D)=\mathcal{R}_l(\mathbf{X}^*)$. Here $\mathcal{C}_r(\mathbf{X})$ is the right column space, and $\mathcal{R}_l(\mathbf{X})$ is the left row space of $\mathbf{X}$. Moreover, $\mathbf{A}^{\textcircled{\dagger}}=\mathbf{A}^k(\mathbf{A}^{k+1})^\dagger$ and $\mathbf{A}_{\textcircled{\dagger}}=(\mathbf{A}^{k+1})^\dagger\mathbf{A}^k$.

Definition 1. [1] Let $\mathbf{A}\in\mathbb{H}^{n\times n}$ with $k=\mathrm{Ind}(\mathbf{A})$. The *right weak group inverse* of $\mathbf{A}$ is the unique solution $\mathbf{X}:=\mathbf{A}^{\textcircled{w}}$ to Eqs. $\mathbf{AX}^2=\mathbf{X}$, $\quad\mathbf{AX}=$

$\mathbf{A}^{\textcircled{\dagger}}\mathbf{A}$. The *left weak group inverse* of $\mathbf{A}$ is the unique solution $\mathbf{X} := \mathbf{A}_{\textcircled{w}}$ to Eqs. $\mathbf{X}^2\mathbf{A} = \mathbf{X}, \quad \mathbf{X}\mathbf{A} = \mathbf{A}\mathbf{A}_{\textcircled{\dagger}}$.

Moreover, $\mathbf{A}^{\textcircled{w}} = \left(\mathbf{A}^{\textcircled{\dagger}}\right)^2 \mathbf{A}$ and $\mathbf{A}_{\textcircled{w}} = \mathbf{A}\left(\mathbf{A}_{\textcircled{\dagger}}\right)^{\mathbf{2}}$.

Definition 2. [2] The *i-th row determinant* of $\mathbf{A} = (a_{ij}) \in \mathbb{H}^{n\times n}$, for any $i \in I_n = \{1, \dots, n\}$, is given by

$$\mathrm{rdet}_i\mathbf{A} := \sum_{\sigma\in S_n} (-1)^{n-r} \left(a_{i\,i_{k_1}} a_{i_{k_1} i_{k_1+1}} \dots a_{i_{k_1+l_1} i}\right) \dots \left(a_{i_{k_r} i_{k_r+1}} \dots a_{i_{k_r+l_r} i_{k_r}}\right),$$

where S_n is the symmetric group on I_n, while the permutation σ is a product of mutually disjoint cycles ordered from the left to right by the rules

$$\begin{gathered}\sigma = \left(i\, i_{k_1} i_{k_1+1} \dots i_{k_1+l_1}\right)\left(i_{k_2} i_{k_2+1} \dots i_{k_2+l_2}\right) \dots \left(i_{k_r} i_{k_r+1} \dots i_{k_r+l_r}\right), \\ i_{k_t} < i_{k_t+s}, \quad i_{k_2} < i_{k_3} < \dots < i_{k_r}, \quad \forall \;\; t = 2, \dots, r, \quad s = 1, \dots, l_t.\end{gathered}$$

For any $j \in I_n$, *the j-th column determinant* of $\mathbf{A}$ is defined as

$$\mathrm{cdet}_j\mathbf{A} = \sum_{\tau\in S_n} (-1)^{n-r} \left(a_{j_{k_r} j_{k_r+l_r}} \cdots a_{j_{k_r+1} j_{k_r}}\right) \cdots \left(a_{j j_{k_1+l_1}} \cdots a_{j_{k_1+1} j_{k_1}} a_{j_{k_1} j}\right),$$

in which a permutation, τ, is ordered from the right to left as follows

$$\begin{gathered}\tau = \left(j_{k_r+l_r} \cdots j_{k_r+1} j_{k_r}\right) \cdots \left(j_{k_2+l_2} \cdots j_{k_2+1} j_{k_2}\right)\;\left(j_{k_1+l_1} \cdots j_{k_1+1} j_{k_1} j\right), \\ j_{k_t} < j_{k_t+s}, \quad j_{k_2} < j_{k_3} < \dots < j_{k_r}.\end{gathered}$$

The non-commutativity of quaternion operations generally results in different $\mathfrak{R}$- and $\mathfrak{C}$-determinants, except in the case where $\mathbf{A}$ is a Hermitian matrix, then

$$\mathrm{rdet}_1\mathbf{A} = \dots = \mathrm{rdet}_n\mathbf{A} = \mathrm{cdet}_1\mathbf{A} = \dots = \mathrm{cdet}_n\mathbf{A} = \alpha \in \mathbb{R}.$$

This property allows us to define the unique determinant of a Hermitian matrix $\mathbf{A}$ by putting $\det\mathbf{A} = \alpha$. The denotation $|\mathbf{A}| := \det\mathbf{A}$ is also used.

The next symbols will be used. Let $\mathbf{a}_{i.}$ and $\mathbf{a}_{.j}$ denote the i-th row and j-th column of $\mathbf{A}$, respectively. Further, $\mathbf{A}_{.j}(\mathbf{c})$ (resp. $\mathbf{A}_{i.}(\mathbf{b})$) stand for the matrix formed by replacing the j-th column (resp. i-th row) of $\mathbf{A}$ by the column vector $\mathbf{c}$ (resp. by the row vector $\mathbf{b}$). Suppose $\alpha := \{\alpha_1, \dots, \alpha_k\} \subseteq \{1, \dots, m\}$ and $\beta := \{\beta_1, \dots, \beta_k\} \subseteq \{1, \dots, n\}$ are subsets with $1 \le k \le \min\{m, n\}$. For $\mathbf{A} \in \mathbb{H}^{m\times n}$, $\mathbf{A}^{\alpha}_{\beta}$ stands for a submatrix with rows and columns indexed by α and β, respectively. When $\mathbf{A} \in \mathbb{H}^{n\times n}$ is Hermitian, $\mathbf{A}^{\alpha}_{\alpha}$ and $|\mathbf{A}|^{\alpha}_{\alpha}$ denote a principal submatrix and a principal minor of $\mathbf{A}$, respectively. Let $L_{k,n} := \{\alpha : \alpha = (\alpha_1, \dots, \alpha_k),\; 1 \le \alpha_1 < \dots < \alpha_k \le n\}$ be the set of strictly increasing sequences of $k \in \{1, \dots, n\}$ integers elected from $\{1, \dots, n\}$. For some $i \in \alpha$ and $j \in \beta$, $I_{r,m}\{i\} := \{\alpha : \alpha \in L_{r,m}, i \in \alpha\}$, $J_{r,n}\{j\} := \{\beta : \beta \in L_{r,n}, j \in \beta\}$.

The following theorem gives determinantal ($\mathfrak{D}$-)representations of the right and left WG inverses.

Theorem 1. [1] *Let* $\mathbf{A} \in \mathbb{H}^{n\times n}$ *with* $k = \operatorname{Ind}(\mathbf{A})$ *and* $\operatorname{rank}(\mathbf{A}^k) = s$. *Then the right WG inverse* $\mathbf{A}^{\circledw} = \left(a_{ij}^{\circledw,r}\right)$ *and the left WG inverse* $\mathbf{A}_{\circledw} = \left(a_{ij}^{\circledw,l}\right)$ *possess the subsequent* $\mathfrak{D}$*-representations*

$$a_{ij}^{\circledw,r} = \frac{\sum\limits_{t=1}^{n} \sum\limits_{\alpha\in I_{s,n}\{t\}} \operatorname{rdet}_t\left(\left[\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right]_{t.}\left(\hat{\mathbf{a}}_{i.}^{(2)}\right)\right)_{\alpha}^{\alpha} a_{tj}}{\sum\limits_{\alpha\in I_{s,n}} \left|\mathbf{A}^{k+2}\left(\mathbf{A}^{k+2}\right)^*\right|_{\alpha}^{\alpha}},$$

$$a_{ij}^{\circledw,l} = \frac{\sum\limits_{t=1}^{n} a_{it} \sum\limits_{\beta\in J_{s,n}\{t\}} \operatorname{cdet}_t\left(\left[\left(\mathbf{A}^{k+2}\right)^*\mathbf{A}^{k+2}\right]_{.t}\left(\check{\mathbf{a}}_{.j}^{(2)}\right)\right)_{\beta}^{\beta}}{\sum\limits_{\beta\in J_{s,n}} \left|\left(\mathbf{A}^{k+2}\right)^*\mathbf{A}^{k+2}\right|_{\beta}^{\beta}},$$

where $\hat{\mathbf{a}}_{i.}^{(2)}$ *is the* i*th row of* $\hat{\mathbf{A}}_2 = \mathbf{A}^k(\mathbf{A}^{k+2})^*$, and $\check{\mathbf{a}}_{.j}^{(2)}$ is the jth column of $\check{\mathbf{A}}_2 = (\mathbf{A}^{k+2})^*\mathbf{A}^k$.

R E F E R E N C E S

1. Kyrchei, I.I., Mosić, D., Stanimirović, P.S. *The right–left WG inverse solutions to quaternion matrix equations*, Symmetry **17**(1), 38 (2025).
2. Kyrchei, I.I. *The theory of the column and row determinants in a quaternion linear algebra.* In: Baswell, A.R. (Ed.), Advances in Mathematics Research 15, pp. 301–359. Nova Sci. Publ., New York (2012).

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REPRESENTATION THEOREMS FOR NONVARIATIONAL SOLUTIONS OF THE HELMHOLTZ EQUATION

We consider a bounded open subset Ω of $\mathbb{R}^n$ of class $C^{1,\alpha}$ for some $\alpha \in]0,1[$ and we plan to present integral representation theorems for α-Hölder continuous solutions of the Helmholtz equation in Ω and in the exterior of Ω that may have an infinite Dirichlet integral around the boundary of Ω. Thus for solutions that do not belong to the classical variational setting.

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SPECIFIC MANIFESTATION OF NONLINEARITIES IN THE PROBLEM OF LIQUID SLOSHING IN RESERVOIRS

The classical theory of nonlinear oscillations is mostly based on the examples of mechanical systems with one degree of freedom. If we transit to the class of mechanical systems with several degrees of freedoms some new aspects of the system behavior appear, however in general few new effects manifest. In the case, when we consider the continuum system, especially the problem of liquid sloshing in reservoirs, we meet considerable complication of the system behavior, which is mostly caused by specific arrangement of the system frequencies. Additional complexity of the system behavior is caused also by the combined character of motion of the liquid and the carrying structure, which corresponds to the main types of practical problems. Here we mention the main specific features of manifestation of nonlinear properties of such kind of the systems with short analysis of mathematical premises of their appearance.

1. In the case of transversal excitation of the motion of the reservoir with a liquid a development of oscillations of antisymmetric modes causes the internal excitation of main symmetrical modes according to nonlinear connection by terms of the second order of smallness. This immediately provide **non-symmetry of waves profiles**, which is supposed to be the main manifestation nonlinearity of the wave propagation [1, 2].

2. Experiments shown that after transversal pulse loading of the carrying structure we observe the **lowering at the midpoint**. Numerical experiments showed that in this case it is necessary to take into account greater than one symmetric mode, which is mostly ignored in other known models [1].

3. Specificity of the liquid sloshing problem consists in the transcendent relation between normal frequencies of separate modes. Since the sum of oscillations with transcendent frequencies is not periodic, it follows that **the problem of periodic motion liquid sloshing has no solution** [4, 5].

4. If we write the solvability condition for the nonlinear sloshing problem? we obtain the condition that liquid must propagate above its unperturbed state within the border, which is specified by reservoir walls. This condition holds automatically for reservoirs of cylindrical shape, however for non-cylindrical reservoirs construction of such kind of solution represent non-trivial problem. The reason is that additional conditions above the liquid free surface contradict with the uniqueness of the classical problem statement for determination of eigenfrequencies and eigenfunctions. However we propose the **method of auxiliary domain** which enables to obtain the required solution [2].

5. Mobility of liquid filling in different structures complicates significantly the problem of control of motion of the carrying body. It is practically impossible to use traditional approaches for the motion control of such systems because of complexity and high dimension of the model of combined motion of liquid with a free surface and the carrying structure. However, the variational approach enables to get in the open form the main vector of interaction of the liquid with reservoir walls (liquid response). Therefore, we propose to use the principle of **compensation of the liquid response** as a control [3]. In this way we eliminate the effect of liquid mobility on the motion of the carrying body. This approach showed its efficiency for linear and nonlinear ranges of the system perturbations. In addition we found that in the case of linear model for the determination of the liquid response efficiency of the control is poor for short time intervals but with further increase of time it becomes better.

6. For problems of forced oscillations of the reservoir with a liquid for both **translational and rotational motion** we observe strong manifestation of **modulation of oscillations**. Near the resonant zone the period of the envelope line is very long, therefore sometimes this type of oscillations is accepted mistakenly as a steady motion [5]. The grater is the exciting frequency, the longer is the period of modulation. In the case of the **system combined motion** its distribution of frequencies considerably depends on the ratio between masses of the liquid and the structure. The effect of combined character of motion results not only in changes of values of frequencies but in changing of the arrangement of frequencies if we put frequencies in the ascending order[3].

7. Since for the combined motion of the carrying structure and the liquid the redistribution of normal frequencies occurs, this causes that the lowest frequency is not correspond to the main normal mode (usually the first antisymmetric mode of oscillations). Therefore, we observe the **manifestation of secondary resonances** [3]. This effect is met mostly for the angular motion of the reservoir on a pendulum suspension with a short suspension lengths when the first normal mode with the circumferential number $m = 2$.

These nonlinear effects are mostly peculiar for the class of problems connected with the combined motion of the reservoirs and a liquid with a free surface. At the same time mainly their manifestation is predetermined by the specific arrangement of the system normal frequencies, when on increase of the number of a frequency the distance between the neighboring frequencies reduces. As the result of this we met transcendent relations between frequencies (which "kills"periodicity of the process) and "grouping"of the highest normal modes.

R E F E R E N C E S

1. Limarchenko, O.S., Matarazzo, G, Yasinsky, V.V. *Rotational motion*

of structu-res with tanks partially filled with liquid, FADA Ltd, Kiev, (2002).
2. Limarchenko, O.S. *Specific features of application of perturbation techniques in problems of nonlinear oscillations of a liquid with free surface in cavities of noncylindrical shape*, Ukr. Math. J. **59**(1), 46-69 (2007).
3. Konstantinov, A.V., Limarchenko, O.S., Lukyanchuk, V.V., Nefedov, A.A. *Dynamic methods of damping the oscillation in structure-free-surface fluid system*, Int. Appl. Mech. **55**(1), 58-67 (2019).
4. Onorato, M., Vozella, L., Proment, D., Lvov, V. *Route to thermalization in the α-Fermi-Pasta-Ulam system* PNAS **112**(14), 4208-4213 (2015).
5. Klimenkov, O.L., Konstantinov, O.V. Limarchenko, O.S. *Vibrations of a Free-Surface Fluid in an Ellipsoidal Tank in the Vicinity Of Resonance*, Int. Appl. Mech., **59**(3), 324-335 (2023).

Łukasz T. Stepien (Krakow, Poland)

ON EXACT SOLUTIONS OF SOME HEAVENLY-LIKE EQUATIONS

Certain exact solutions (the so-called functionally-invariant solutions) of some heavenly-like equations will be presented.

R E F E R E N C E S

1. Stepien, Ł. T. *On some exact solutions of heavenly equations in four dimensions*, *AIP Advances* **10**, 065105 (2020); arXiv:1208.2905 (2012).

Sergiy Maksymenko (Kyiv, Ukraine)

DEFORMATIONS OF SMOOTH FUNCTIONS ON COMPACT SURFACES

Let M be a smooth compact surface. Denote by $\mathcal{F}(M)$ the set of C^∞ functions $f\colon M \to \mathbb{R}$ satisfying the following two conditions:

(a) f has no critical points on the boundary ∂M and takes constant values on its connected components;

(b) at each critical points p of f there are local coordinates (x, y) in which f is a homogeneoous polynomial without multiple factors.

For instance, if $g\colon U \to \mathbb{C}$ is a holomorphic function defined on some open subset $U \subset \mathbb{C}$, then the critical points of its real and imaginary part, $\Re(g)$ and $\Im(g)$, satisfy condition (b).

Let $\mathcal{D}(M)$ be the group of C^∞ diffeomorphisms of M. Then there is a natural right action of $\mathcal{D}(M)$ on the space $C^\infty(M)$ of C^∞ function $M \to \mathbb{R}$ defined by

$$\mu\colon C^\infty(M) \times \mathcal{D}(M) \to C^\infty(M), \qquad \mu(f,h) := f \circ h.$$

For $f \in C^\infty(M)$ let

$$\mathcal{S}(f) := \{h \in \mathcal{D}(M) \mid f \circ h = f\}, \qquad \mathcal{O}(f) := \{f \circ h \mid h \in \mathcal{D}(M)\},$$

be the corresponding stabilizer and orbit of f with respect to the above action.

Let $\Delta_f = \{f^{-1}(c) \mid c \in \mathbb{R}\}$ the partition of M into level sets of f. Then the partition $\Delta_{f\circ h} = \{h^{-1}(f^{-1}(c)) \mid c \in \mathbb{R}\}$ for the function $f \circ h$ consists of inverse images under h of elements of Δ_f. In particular, $\mathcal{S}(f)$ consists of diffeomorphisms $h\colon M \to M$ which leave invariant the elemets of Δ_f.

The aim of the talk is to describe a recent progres in the computations of the homotopy types (deformational properties) of $\mathcal{S}(f)$ and $\mathcal{O}(f)$.

R E F E R E N C E S

1. Kuznietsova, I., Maksymemko, S. *Deformational symmetries of smooth functions on non-orientable surfaces*, Topological Methods in Nonlinear Analysis **65**(1), 35-77 (2025)

2. Maksymenko, S. *Deformations of functions on surfaces by isotopic to the identity diffeomorphisms*, Topology and its Applications, **282**, 107312 (2020)

Agnieszka Niemczynowicz (Kraków, Poland)

HYPERCOMPLEX NEURAL NETWORKS: CURRENT STATE OF KNOWLEDGE AND FUTURE PERSPECTIVES

Hypercomplex-valued neural networks (HvNNs) represent a rapidly evolving field that extends traditional artificial neural networks into higher-dimensional number systems such as complex numbers, quaternions, and octonions. These architectures offer unique advantages in handling multidimensional data, preserving phase and amplitude information, and reducing network complexity through parameter sharing.

This lecture provides a comprehensive overview of the current state of knowledge on hypercomplex-valued neural networks, covering their mathematical foundations, architectural designs, and key application areas. We also examine the main challenges facing researchers in this domain and highlight the growing potential of HvNNs as a powerful tool in modern machine learning, while emphasizing the need for continued theoretical and practical advancements.

R E F E R E N C E S

1. Dickson, L. E. *On quaternions and their generalization and the history of the eight square theorem*, Annals of Mathematics, 20(3), 155—171, 1919.
2. Cayley, A., *On certain results relating to quaternions,* Philosophical Magazine 26 141—145 (1845).
3. Reprinted in pp. 123—126 of The Collected Mathematical Papers of Arthur Cayley, Sc.D., F.R.S., Vol. 1, Cambridge University Press, Cambridge (1889).
4. Cybenko, G. (1989). *Approximation by superpositions of a sigmoidal function.* MCSS, 2(4), 303—314.
5. Hornik, K., Stinchcombe, M., & White, H. (1989). *Multilayer feedforward networks are universal approximators.* Neural Networks, 2(5), 359—366.
6. Niemczynowicz, A., & Kycia, R. A. (2025). *KHNN: Hypercomplex-valued neural networks computations via Keras using TensorFlow and PyTorch.* SoftwareX, 30, 102163. `https://doi.org/10.1016/j.softx.2025.102163`

Małgorzata Nowak-Kępczyk (Lublin, Poland)

MODELING MATHEMATICS STUDENT MOTIVATION SHIFTS THROUGH PSYCHOLOGICAL TEACHING SPACES

Like a boat we built with trust — sleek, fast, and seemingly full of promise — modern educational change, propelled by ever-present media and AI usage, carries our children swiftly forward. We believed this was the best course. But in its speed and shine, this vessel may be drifting away from something essential: the quiet, deliberate, deeply human art of teaching.

We are beginning to see the cost — in students' ability to focus, to reflect, and to think with depth and independence.

Our study, based on a questionnaire completed by 183 participants across age groups, investigates the motivational space created by math teachers and how it influences both short- and long-term student engagement.

We identify characteristic teaching styles and, using a modified Principal Component Analysis (PCA), model how student motivation shifts as educational environments evolve.

From this, we propose a novel four-dimensional geometric model — a dynamic tetrahedron — to visualize these motivational transformations and trace students' trajectories through different learning contexts.

This approach highlights the teacher's central role in shaping not only learning outcomes, but also the psychological and cognitive foundations of education. It offers a practical, transferable framework for understanding and simulating how teaching practices affect student development — an urgently needed compass in the journey through our times.

R E F E R E N C E S

1. Cipora, K., Lyons, A. M., Ansari, E. E., Szczygiel, M. A., and Dowker, E. E. *Mathematics anxiety: Insights from psychology, neuroscience and education.* Trends in Neuroscience and Education, 28:100170 (2022).

3. Fraser, B. J. Classroom Environment. Croom Helm, London (1986).

4. Nowak-Kępczyk, M. *In the Space a Teacher Creates: Modeling Student Movement Across Psychological Landscapes*, preprint (2025)

5. Nowak-Kępczyk, *The Forgotten Dimension: Personal Support Predicts Shifts in Math Motivation Profiles*, preprint (2025)

5. OECD (2021). *The state of global education: 18 months into the pandemic.* www.oecd.org/education/state-of-global-education-2021.html. Organisation for Economic Co-operation and Development.

6. Pianta, R. C., Hamre, B. K., and Allen, J. P. *Teacher-student relationships and engagement: Conceptualizing, measuring, and improving the capacity of classroom interactions.* In Christenson, S. L., Reschly, A. L., and Wylie, C., editors, Handbook of Research on Student Engagement, Springer, Boston, MA, pages 365-386 (2012).

7. Skinner, E. A., Kindermann, J. R., and Furrer, A. A motivational perspective on engagement and disaffection: Conceptualization and assessment of childrenвЂ™s behavioral and emotional participation in academic activities in the classroom. Educational and Psychological Measurement, 69(3):493-525 (2009).

8. UNESCO Education: From disruption to recovery. https://www.unesco.org/ en/covid-19/education-response. United Nations Educational, Scientific and Cultural Organization, (2021).

9. Wijaya, A., Kurniawan, D., and Rahmawati, Y. *Analysis of factors affecting wellbeing and academic performance of doctoral students in mathematics education in Indonesia.* International Journal of STEM Education, 10:13, (2023).

Tetiana Osipchuk (Kyiv, Ukraine)

ON WEAKLY 1-CONVEX SETS IN EUCLIDEAN SPACES

Weakly 1-convex sets in the n-dimensional real Euclidean space, $\mathbb{R}^n$, $n \geqslant 2$, can be considered as a real analogue of lineally convex sets of multidimensional complex space. The notion was coined by Yurii Zelinskii [1].

An open set is called ***weakly* 1*-convex*** if for any boundary point of the set there exists a straight line passing through this point and not intersecting the given set [1]. A closed set is called ***weakly* 1*-convex*** if it is approximated from the outside by a family of open weakly 1-convex sets [1]. A point of the complement of a set to the whole space is called a **1*-nonconvexity point*** of the set if any straight line passing through the point intersects the set. The collection of all 1-nonconvexity points of a subset $E \subset \mathbb{R}^n$ is said to be the **1*-nonconvexity-point set*** corresponding to E and is denoted by $E^{\triangle}$ [2].

Theorem 1 (Dakhil [3], Osipchuk [4])**.** *An open or closed weakly* 1*-convex subset* $E \subset \mathbb{R}^2$ *with non-empty* 1*-nonconvexity-point set consists of not less than three connected components.*

Theorem 2 (Osipchuk [4])**.** *There exist weakly* 1*-convex domains and closed connected subsets of* $\mathbb{R}^n$*,* $n \geqslant 3$*, with non-empty* 1*-nonconvexity-point sets.*

Theorem 3 (Osipchuk [5, 6])**.** *Let a subset* $E \subset \mathbb{R}^2$ *be open and weakly* 1*-convex, and let* $E^{\triangle} \neq \varnothing$*. Let* $E_j^{\triangle}$*,* $j \in N \subseteq \mathbb{N}$*, be the components of* $E^{\triangle}$*. Then*

(a) $E^{\triangle}$ *is open and weakly* 1*-convex;*

(b) *each component* $E_j^{\triangle}$*,* $j \in N$*, is convex;*

(c) *any connected part of* $\partial E_j^{\triangle}$*,* $j \in N$*, consisting of only smooth points is either a line segment, a ray, or a point;*

(d) *there exists a collection of straight lines* $\{L^k\}_{k \in M}$*,* $M \subseteq \mathbb{N}$*, such that*

- $\bigcup\limits_k L^k \supset \partial E^{\triangle}$*,*
- *the set* $\bigcup\limits_k L^k \bigcup E^{\triangle}$ *does not contain straight lines intersecting* $E^{\triangle}$*,*
- $\bigcup\limits_k L^k \bigcap E = \varnothing$*;*

(e) *each component* $E_j^{\triangle}$*,* $j \in N$*, is the interior of a convex polygon, if* E *is bounded and consists of finite number of components.*

Theorem 4 (Osipchuk [7]). *Suppose that an open subset $E \subset \mathbb{R}^n$ is weakly* 1*-convex with non-empty set $E^\triangle$. Then $E^\triangle$ is open and weakly* 1*-convex, and there exists a collection of straight lines $\{L(x)\}_{x\in\partial E^\triangle}$ passing through the points $x \in \partial E^\triangle$ such that*

- *the set $\bigcup\limits_{x\in\partial E^\triangle} L(x) \bigcup E^\triangle$ does not contain straight lines intersecting $E^\triangle$,*
- $\bigcup\limits_{x\in\partial E^\triangle} L(x) \bigcap E = \varnothing$.

Theorem 5 (Osipchuk [7]). *There exists an open weakly* 1*-convex domain $E \subset \mathbb{R}^n$, $n \geqslant 3$, such that $E^\triangle$ is non-empty, bounded (or unbounded), connected, and non-convex.*

Theorem 6 (Osipchuk [7]). *Let $E \subset \mathbb{R}^n$ be a closed subset such that* $\operatorname{Int} E \neq \varnothing$*. If E is weakly* 1*-convex, then* $\operatorname{Int} E$ *is weakly* 1*-convex.*

R E F E R E N C E S

1. Momot, I.V., Zelinskii, Yu.B. *On (n, m)-convex sets*, Ukr. Mat. Zhurnal, **53**(3), 422–427 (2001).

2. Osipchuk, T.M. *Topological properties of weakly m-convex sets*, Proceedings of IAMM of NAS of Ukraine, **34**, 75–84 (2020).

3. Dakhil, H.K. *The shadows problems and mappings of fixed multiplicity*, candthesis, Institute of Mathematics of NAS of Ukraine, Kyiv, (2017).

4. Osipchuk, T.M. *On closed weakly m-convex sets*, Proceedings of the International Geometry Center, **15**(1), 50–65 (2022).

5. Osipchuk, T.M. *Topological and Geometric Properties of the Set of* 1*-Nonconvexity Points of a Weakly* 1*-Convex Set in the Plane*, Ukrainian Mathematical Journal, **73**(12), 1918–1936 (2022).

6. Osipchuk, T.M., Tkachuk, M.V. *On weakly* 1*-convex sets in the plane*, Proceedings of the International Geometry Center, **16**(1), 42–49 (2023).

7. Osipchuk, T.M. *On weakly* 1*-convex and weakly* 1*-semiconvex sets in real Euclidean spaces*, Proceedings of the International Geometry Center, **18**(1), 102–117 (2025).

Dariusz Partyka (Lublin, Poland)

GEOMETRIC PROPERTIES OF HARMONIC FUNCTIONS IN THE UNIT DISC WITH BOUNDARY NORMALIZATION

Given $n \in \mathbb{N}$ let $T_1, T_2, \ldots, T_n$ be closed arcs contained in the unit circle $\mathbb{T} := \{z \in \mathbb{C} : |z| = 1\}$, with positive length, total length 2π and covering $\mathbb{T}$. Write

$\mathcal{F}$ for the class of all complex-valued harmonic functions F of the unit disc $\mathbb{D} := \{z \in \mathbb{C} : |z| < 1\}$ into itself satisfying the following sectorial condition: For each $k \in \{1, 2, \ldots, n\}$ and for almost every $z \in T_k$ the radial limit of F at the point z belongs to the angular sector determined by the convex hull spanned by the zero and arc T_k. Another class $\mathcal{H}$ under consideration consists of all harmonic diffeomorphisms F of $\mathbb{D}$ onto itself satisfying the following classical boundary condition: For every $k \in \{1, \ldots, n\}$ the radial limit of F at the point e_k is equal to e_k. Here $\{e_1, e_2, \ldots, e_n\}$ is a subset of $\mathbb{T}$ containing n points placed according to the positive orientation of $\mathbb{T}$. The talk is a survey of results on geometric properties for both classes, obtained in cooperation with Anna Futa and Józef Zając in [1]–[5]. An example result is the following Schwarz type inequality

$$|F(z)| \leq \frac{2}{\pi} \arctan\left(\sqrt{3}\frac{1+|z|}{1-|z|}\right), \quad F \in \mathcal{F},\ z \in \mathbb{D}, \tag{1}$$

where the class $\mathcal{F}$ is defined by the arcs $T_k := \{e^{it} \colon t \in [2\pi(k-1)/3; 2\pi k/3]\}$, $k \in \{1, 2, 3\}$; cf. [1, Corollary 2.2 and Remark 2.5]. The inequality (1) is also valid for the class $\mathcal{H}$ defined by the points e_1, e_2 and e_3 being successive cube roots of unity; cf. [1, Corollary 2.4 and Remark 2.5]. The sets $\{F(0) \colon F \in \mathcal{F}\}$ and $\{F(0) \colon F \in \mathcal{H}\}$ are precisely determined in [3] and [5], respectively. The remaining presented inequalities related to (1) for the more general case of the classes $\mathcal{F}$ and $\mathcal{H}$ are taken from [2] and [4].

R E F E R E N C E S

1. Partyka, D., Zając, J., *The Schwarz type inequality for harmonic mappings of the unit disc with boundary normalization*, Complex Anal. Oper. Theory **9**, 213–228 (2015).
2. Futa, A., Partyka, D., *The Schwarz type inequality for harmonic functions of the unit disc satisfying a sectorial condition*, Bull. Soc. Sci. Lettres Łódź **68**, no. 2, 97–109 (2018); Série: Recherches sur les déformations.
3. Partyka, D., Zając, J., *The range set of zero for harmonic mappings of the unit disc with sectorial boundary normalization*, Ann. Fenn. Math. **49**, 49–63 (2024).
4. Futa, A., Partyka, D., *Schwarz type inequalities for harmonic functions of the unit disk and normalized on the boundary* (unpublished) (2025).
5. Partyka. D., Zając, J., *The range set of zero for boundary normalized harmonic diffeomorphisms of the unit disc* (unpublished) (2025).

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EXTREMAL PROBLEMS ON THE MINIMIZATION OF THE AREA FUNCTIONAL

Let G be a domain in the complex plane $\mathbb{C}$. A Sobolev homeomorphism $f \in W^{1,1}_{\mathrm{loc}}$ is termed *regular* in G if $J_f > 0$ almost everywhere in G.

We will use the following notation:

$$B_r = \{z \in \mathbb{C} : |z| < r\}\,, \quad \mathbb{B} = \{z \in \mathbb{C} : |z| < 1\}\,, \quad \gamma_r = \{z \in \mathbb{C} : |z| = r\}\,.$$

For a regular Sobolev homeomorphism $f : \mathbb{B} \to \mathbb{C}$ and $p > 1$, we define the *p-angular dilatation* at $z = re^{i\theta} \in \mathbb{B} \setminus 0$ with respect to $z_0 = 0$ as:

$$D_{p,f}(z) = \frac{|f_\theta(re^{i\theta})|^p}{r^p J_f(re^{i\theta})}\,.$$

For $D_{p,f}(z)$ and $r \in (0,1)$, we denote

$$d_{p,f}(r) = \left(\frac{1}{2\pi r} \int\limits_{\gamma_r} D_{p,f}^{\frac{1}{p-1}}(z)\,|dz| \right)^{p-1}\,. \tag{1}$$

Consider the area functional $S_r(f) = |f(B_r)|$. We first fix $1 < p < 2$, $K > 0$, and $\alpha < 2 - p$, and define $\mathcal{H}_1 = \mathcal{H}_1(p, K, \alpha)$ as the set of all regular Sobolev homeomorphisms $f : \mathbb{B} \to \mathbb{C}$ (in $W^{1,1}_{\mathrm{loc}}$) possessing the Lusin (N)-property and satisfying the condition

$$d_{p,f}(t) \leqslant K t^\alpha$$

for a.e. (almost every) $t \in [0,1)$.

Theorem 1. *For all $r \in [0,1)$, the following equality holds:*

$$\min_{f \in \mathcal{H}_1} S_r(f) = \pi\, K^{-\frac{2}{2-p}} \left(\frac{2-p}{2-p-\alpha} \right)^{\frac{2}{2-p}} r^{\frac{2(2-p-\alpha)}{2-p}}\,.$$

Moreover, the minimum of this functional is attained by the radial stretching

$$\tilde{f}(z) = \begin{cases} K^{-\frac{1}{2-p}} \left(\frac{2-p}{2-p-\alpha} \right)^{\frac{1}{2-p}} |z|^{-\frac{\alpha}{2-p}}\, z, \ z \neq 0, \\ 0, \ z = 0. \end{cases}$$

Next, we consider the case with the mixed growth condition. Fix $1 < p < 2$, $K > 0$, and $\beta > 1$, and denote by $\mathcal{H}_2 = \mathcal{H}_2(p, K, \beta)$ the set of all regular Sobolev homeomorphisms $f : \mathbb{B} \to \mathbb{C}$ (in $W^{1,1}_{\mathrm{loc}}$) that preserve the Lusin (N)-property and satisfy the mixed growth condition

$$d_{p,f}(t) \leqslant K t^{2-p} \left(\ln \frac{e}{t}\right)^{\beta}$$

for a.e. $t \in [0, 1)$.

Theorem 2. *For all $r \in [0, 1)$, the following equality holds:*

$$\min_{f \in \mathcal{H}_2} S_r(f) = \pi K^{-\frac{2}{2-p}} \left(\frac{2-p}{\beta - 1}\right)^{\frac{2}{2-p}} \left(\ln \frac{e}{r}\right)^{\frac{2(1-\beta)}{2-p}}.$$

Moreover, the minimum of this functional is attained by

$$\hat{f}(z) = \begin{cases} K^{-\frac{1}{2-p}} \left(\frac{2-p}{\beta-1}\right)^{\frac{1}{2-p}} \left(\ln \frac{e}{|z|}\right)^{\frac{1-\beta}{2-p}} \frac{z}{|z|}, \ z \neq 0, \\ 0, \ z = 0. \end{cases}$$

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Sergiy Plaksa (Kyiv, Ukraine)

CONTINUOUS EXTENSION OF THE LOGARITHMIC DOUBLE LAYER POTENTIAL TO THE ALPHORS-REGULAR BOUNDARY OF A DOMAIN

Let γ be a closed rectifiable Jordan curve in the complex plane $\mathbb{C}$, and let D^+ and D^- be the interior and exterior domains bounded by γ, respectively.

We consider the logarithmic double layer potential

$$\frac{1}{2\pi} \int_{\gamma} g(t) \frac{\partial}{\partial \mathbf{n}_t} \left(\ln \frac{1}{|t - z|}\right) ds_t = \mathrm{Re} \left(\frac{1}{2\pi i} \int_{\gamma} \frac{g(t)}{t - z} \, dt\right) \qquad \forall\, z \in D^{\pm},$$

where $\mathbf{n}_t$ and s_t denote the unit vector of the outward normal to the curve γ at a point $t \in \gamma$ and an arc coordinate of this point, respectively, and the integral density $g : \gamma \to \mathbb{R}$ takes values in the set of real numbers $\mathbb{R}$.

The classical theory of the logarithmic double layer potential is developed in the case where γ is a Lyapunov curve (see J. Plemelj [1]) or a Radon curve of bounded rotation (see J. Radon [2]). J. Král [3] established a necessary and sufficient condition for the curve γ, under which the logarithmic double layer potential is continuously extended from the domains $D^{\pm}$ to the boundary for all continuous functions g.

We consider an *Ahlfors-regular* curve (see [4, 5]), i.e., a closed rectifiable Jordan curve γ satisfying the condition $\sup\limits_{\xi\in\gamma} \operatorname{mes}\gamma_{\varepsilon}(\xi) = O(\varepsilon)$, $\varepsilon \to 0$, where $\gamma_{\varepsilon}(\xi) := \{t \in \gamma : |t-\xi| \le \varepsilon\}$ and mes denotes the linear Lebesgue measure on the curve γ.

The class of Ahlfors-regular curves includes as a subclass the curves from the mentioned Král's result (as well as the Lyapunov curves and the Radon curves).

For the logarithmic double layer potential, a necessary and sufficient condition for the continuous extension to the Ahlfors-regular boundary is established in the following

Theorem. *Let a closed Jordan curve γ be Ahlfors-regular and let a function $g\colon \gamma \to \mathbb{R}$ be continuous on γ. The logarithmic double layer potential is continuously extended to the boundary γ from the domain D^{+} or D^{-} if and only if the following condition is satisfied:*

$$\sup_{\xi\in\gamma}\ \sup_{\delta\in(0,\varepsilon)} \left| \int\limits_{\gamma_{\varepsilon}(\xi)\setminus\gamma_{\delta}(\xi)} \big(g(t)-g(\xi)\big)\, d\arg(t-\xi) \right| \to 0\,, \qquad \varepsilon \to 0\,.$$

In the case where this condition is satisfied, for all $\xi \in \gamma$, the limiting values of the logarithmic double layer potential from the domain $D^{\pm}$ are represented by the formulas

$$\lim_{z\to\xi, z\in D^{+}} \operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)}{t-z}\,dt\right) = g(\xi) + \frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\,,$$

$$\lim_{z\to\xi, z\in D^{-}} \operatorname{Re}\left(\frac{1}{2\pi i}\int\limits_{\gamma}\frac{g(t)}{t-z}\,dt\right) = \frac{1}{2\pi}\int\limits_{\gamma}\big(g(t)-g(\xi)\big)\, d\arg(t-\xi)\,.$$

For the continuous extension of the logarithmic double layer potential to the boundary of a domain, we also consider sufficient conditions that cover the case of curves of a wider class than in the mentioned Kral's result. Illustrative examples will be presented.

R E F E R E N C E S

1. Plemelj, J. *Potentialtheoretische Untersuchungen*, Teubner, Leipzig (1911).

2. Radon, J. *Über die randwertaufgaben beim logarithmische potential*, Sitz. Akad. Wiss. Wien, Band **12**, Heft **7**, 1123–1167 (1919).

3. Král, J. *On the logarithmic potential of the double distribution*, Czech, math. J. **14** (2), 306–321 (1964).
4. Ahlfors, L. *Zur theorie der überlagerungsflächen*, Acta Math. **65**, 157–194 (1935).
5. Salaev, V.V. *Direct and inverse estimates for a singular Cauchy integral along a closed curve*, Mathematical Notes of the Academy of Sciences of the USSR **19** (3), 221–231 (1976).

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RIEMANN ZETA FUNCTION OF A VARIABLE FROM COMMUTATIVE ALGEBRA

Abstract. We introduce and study the extension of the Riemann zeta function to commutative algebras with identity.

Suppose $\mathbb{A}$ is a d-dimensional commutative algebra over a field $\mathbb{K}=\mathbb{R}$ (or $\mathbb{C}$) with a basis e_1, e_2,…, e_d and e_1 is the identity of $\mathbb{A}$.

Suppose $\mathbb{A}$ has non-trivial idempotents i_1, i_2,…, i_d such that $i_k i_m = 0$, $k \neq m$, and $\sum_{k=1}^{d} i_k = 1$.

Denote by $I_m = \{a i_m | a \in \mathbb{A}\}$ the principal ideal generated by i_m, $m = 1, \ldots, d$. It is easily seen that $\mathbb{A}$ can be decomposed in the direct sum (the Pierce decomposition)

$\mathbb{A} = I_1 \oplus I_2 \oplus \ldots \oplus I_d$.

Lemma 1. *Idempotents i_1, i_2,…, i_d are linearly independent.*

Lemma 2. *If $b \in I_m$ then there exists $k \in \mathbb{K}$ such that $b = k i_m$, i.e., the ideal I_m can be represented as follows $I_m = \{k i_m | k \in \mathbb{K}\}$.*

Theorem 1. *For any $a \in \mathbb{A}$ we have the following decomposition*

$$a = \sum_{m=1}^{d} k_m i_m, \quad k_m \in \mathbb{K}.$$

It is easily verified that $a^l = \sum_{m=1}^{d} k_m^l i_m$.

Let us define $a \in \mathbb{A}$ power of a natural number.

For $a = \sum_{m=1}^{d} k_m i_m$ and $n \in \mathbb{N}$, we have

$$n^a = \exp\left(a \ln n\right) = \sum_{l=0}^{\infty} \frac{a^l \left(\ln n\right)^l}{l!} = \sum_{m=1}^{d} \sum_{l=0}^{\infty} \frac{k_m^l \left(\ln n\right)^l}{l!} i_m = \sum_{m=1}^{d} n^{k_m} i_m.$$

The zeta function of $a \in \mathbb{A}$ variable

For $a = \sum_{m=1}^{d} k_m i_m$

$$\zeta(a) = \sum_{n=0}^{\infty} n^{-a} = \sum_{n=0}^{\infty} \sum_{m=1}^{d} n^{-k_m} i_m = \sum_{m=1}^{d} \sum_{n=0}^{\infty} n^{-k_m} i_m = \sum_{m=1}^{d} \zeta\left(k_m\right) i_m.$$

Conclusion.

If $\mathbb{K} = \mathbb{R}$ the zeta function has only the trivial zeros at points $k_m = -2l_m$, $l_m \in \mathbb{N}$, $m = 1, 2, \ldots, d$.

If $\mathbb{K} = \mathbb{C}$ the $\mathbb{A}$ algebra Riemann hyposesis is as follows:

Non-trivial zeros of the zeta function of variable $a \in \mathbb{A}$ are located at $a = \frac{1}{2} + i\Sigma$, where $\Sigma = \sum_{m=1}^{d} \sigma_m i_m$, $m = 1, 2, \ldots, d$ and $\frac{1}{2} + i\sigma_m$ is zero of the complex zeta function $\zeta(z)$.

R E F E R E N C E S

1. Rochon D., A Bicomplex Riemann Zeta Function. Tokyo J.Math. Vol. 27, No. 2, 2004.

2. Reid F. L., Van Gorder, A Multicomplex Riemann Zeta Functions. Adv. Appl. Clifford, 23, Iss. 1, (2003), 237-251.

3. Pogorui A., Hyperholomorphic Functions in Commutative Algebras. Complex variables and elliptic equations, Vol. 52, Iss. 12, (2007), 1155 - 1159.

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ONCE AGAIN ABOUT THE EXCEPTIONAL SET IN ERDŐS-MACINTYRE TYPE THEOREM'S FOR DIRICHLET SERIES AND SOME CONJECTURES

Denote by $D(\Lambda)$ the class of entire (absolutely convergent in the complex plane) Dirichlet series of the form $F(z) = \sum_{n=0}^{+\infty} a_n e^{z\lambda_n}$, where $\Lambda = (\lambda_n)$ is a fixed sequence such that $0 = \lambda_0 < \lambda_n \uparrow +\infty$ $(1 \leq n \to +\infty)$.

Let us introduce some notations for $F \in D(\Lambda)$ and $x \in \mathbb{R}$: $\mu(x, F) = \max\{|a_n|e^{x\lambda_n} : n \geq 0\}$ is the maximal term, $M(x, F) = \sup\{|F(x+iy)| : y \in \mathbb{R}\}$ is the maximum modulus, $m(x, F) = \inf\{|F(x + iy)| : y \in \mathbb{R}\}$ is the minimum modulus, $\nu(x, F) = \max\{n : |a_n|e^{x\lambda_n} = \mu(x, F)\}$ is the central index of the Dirichlet series.

O.B. Skaskiv (1984) proved the following theorem.

Theorem A (O.B. Skaskiv, 1984). *For every entire function $F \in D(\Lambda)$ the relation*

$$F(x+iy) = (1+o(1))a_{\nu(x,F)}e^{(x+iy)\lambda_{\nu(x,F)}} \qquad (1)$$

holds as $x \to +\infty$ outside some set E of finite Lebesgue measure ($\int_E dx < +\infty$) uniformly in $y \in \mathbb{R}$, if and only if

$$\sum_{n=0}^{+\infty} \frac{1}{\lambda_{n+1}-\lambda_n} < +\infty. \qquad (2)$$

The finiteness of Lebesgue measure of an exceptional set E in Theorem A is the sharp estimate in the whole class of entire functions $F \in D(\Lambda)$ (T.M. Salo, O.B. Skaskiv, Mat.Stud., 2001). Therefore, the problem of the sharp describing the magnitude of the exceptional set in relation (1) for an arbitrary given function $F \in D(\Lambda)$ such that the exponents Λ satisfy condition (2) naturally arises. Such a problem was first formulated by I.V. Ostrovskii (1995) regarding the exceptional set in the classical Wiman inequality for entire functions represented by power series.

Let Φ be apositive increasing to $+\infty$ continuos function on $[0, +\infty)$, φ be the inverse function to function Φ and h be a differentiable on $[0, +\infty)$ function such that $h'(x) \searrow +\infty$ $(x \searrow +\infty)$. Consider the class

$$D(\Lambda, \Phi) = \{F \in D(\Lambda)\colon\ (\exists K_j > 0) :\ \ln \mu(x, \Phi) \geq K_1 x \Phi(K_2 x) \quad (x > x_0)\}.$$

T.Salo and O.Skaskiv proved ([1, 2]) the following theorem.

Theorem 1. *For the asymptotic relation (1) to hold for every function $F \in D(\Lambda, \Phi)$ as $x \to +\infty$ outside some set E of finite h-measure, i.e. $\int_E dh(x) < +\infty$, uniformly in $y \in \mathbb{R}$, it is necessary and sufficient that*

$$(\forall b > 0) :\ \sum_{n=0}^{+\infty} \frac{h'(b\varphi(b\lambda_n))}{\lambda_{n+1}-\lambda_n} < +\infty.$$

Conjecture 1. *The description of an exeptional set E in Theorem 1 is the best possible in the class $D(\Lambda, \Phi)$.*

Let us cosider the class

$$D_\varphi(\Lambda) = \big\{F \in D(\Lambda)\colon\ (\exists n_0)(\forall n \geq n_0)[|a_n| \leq \exp\{-\lambda_n \varphi(\lambda_n)\}]\big\}.$$

Conjecture 2. *Let φ be positive increaing to $+\infty$ continuos function, and h be positive decreasing function such that $h'(x) \searrow 0$ $(x \nearrow +\infty)$. If*

$$\sum_{n=0}^{+\infty} \frac{h'(\varphi(\lambda_n))}{\lambda_{n+1}-\lambda_n} < +\infty,$$

then for each $F \in D_\varphi(\Lambda)$ relation (1) holds as $x \to +\infty$ outside some set E of finite h-measure uniformly in $y \in \mathbb{R}$.

R E F E R E N C E S

1. Salo, T.M., Skaskiv, O.B., *Minimum modulus of lacunary power series and h-measure of exceptional sets*, Ufa Math. J., **9** (2017)(4), 135–144 (2017). doi:10.13108/2017-9-4-135

2. Salo, T.M., Skaskiv, O.B., *The minimum modulus of gap power series and h-measure of exceptional sets*, arXiv: 1512.05557v1 [math.CV] 17 Dec 2015, 13 p.

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ON EXISTENCE OF SOLUTIONS OF BELTRAMI EQUATIONS IN THE CONTEXT OF TANGENTIAL DILATATION

Let $\mu=\mu(z,w):D\times\mathbb{C}\rightarrow\mathbb{D}$ be $\nu=\nu(z,w):D\times\mathbb{C}\rightarrow\mathbb{D}$ some functions. Let us consider the equation

$$f_{\overline{z}}=\mu(z,f(z))\cdot f_z+\nu(z,f(z))\cdot\overline{f_z}\,, \tag{1}$$

where $f_{\overline{z}}=(f_x+if_y)/2$ and $f_z=(f_x-if_y)/2$. Fix $n\geqslant 1$ and set

$$\mu_n(z,w)=\begin{cases}\mu(z,w), & K_{\mu,\nu}\leqslant n,\\ 0\,, & K_{\mu,\nu}>n\,,\end{cases}$$

and

$$\nu_n(z,w)=\begin{cases}\nu(z,w), & K_{\mu,\nu}\leqslant n,\\ 0\,, & K_{\mu,\nu}>n\,,\end{cases}$$

where $K_{\mu,\nu}(z,w)=\frac{1+|\mu(z,w)|+|\nu(z,w)|}{1-|\mu(z,w)|-|\nu(z,w)|}$. Assume that, $\mu=\mu(z,w):D\times\mathbb{C}\rightarrow\mathbb{D}$ and $\nu=\nu(z,w):D\times\mathbb{C}\rightarrow\mathbb{D}$ satisfy Caratheodory conditions, i.e., ν is measurable by $z\in D$ for all fixed $w\in\mathbb{C}$, and continuous by w for almost all $z\in D$. Now

$$K_{\mu_n,\nu_n}(z,w)=\frac{1+|\mu_n(z,w)|+|\nu_n(z,w)|}{1-|\mu_n(z,w)|-|\nu_n(z,w)|}\leqslant n$$

for almost all $z\in D$ and all $w\in\mathbb{C}$. Now the equation $f_{\overline{z}}=\mu_n(z,f(z))\cdot f_z+\nu_n(z,f(z))\cdot\overline{f_z}$ has a homeomorphic solution f_n in $W_{\rm loc}^{1,1}(D)$ such that $f_n^{-1}\in W_{\rm loc}^{1,2}(f_n(D))$ and $f_n(0)=0$, $f_n(1)=1$.

Let f_n be a solution of (1) and $g_n = f_n^{-1}$. Set $K^T_{\mu_{g_n}}(w,w_0) = \frac{\left|1-\frac{\overline{w-w_0}}{w-w_0}\mu_{g_n}(w)\right|^2}{1-|\mu_{g_n}(w)|^2}$ and $K_{I,p}(w,g_n) = \frac{|(g_n)_w|^2-|(g_n)_{\overline{w}}|^2}{(|(g_n)_w|-|(g_n)_{\overline{w}}|)^p}$.

Theorem. *Let μ, ν, μ_n, ν_n, f_n and g_n be defined as above. Let $Q, Q_0 : \mathbb{C} \to [0,\infty]$ ne Lebesgue measurable functions. Assume that $K_{\mu,\nu}(z,w) \leqslant Q_0(z) < \infty$ for all $w \in \mathbb{C}$ and almost all $z \in D$. Assume that the following conditions hold:*

1) for all $0 < r_1 < r_2 < 1$ and $y_0 \in \mathbb{C}$ there is $E \subset [r_1, r_2]$ of a positive Lebesgue measure such that Q is integrable over $S(y_0, r)$ for all $r \in E$;

2) there are $1 < p \leqslant 2$ and $M > 0$ such that $\int\limits_{f_n(D)} K_{I,p}(w,g_n)\,dm(w) \leqslant M$ for all $n = 1, 2, \ldots$, where $K_{I,p}(w,g_n)$ is defined as above;

3) the relation

$$K^T_{\mu_{g_n}}(w,w_0) \leqslant Q(w)$$

holds for all $w \in f_n(D)$ and all $w_0 \in f_n(D)$, where $K^T_{\mu_{g_n}}$ is defined above. Then the equation (1) has a continuous $W^{1,p}_{\rm loc}(D)$-solution f in D.

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σ-MONOGENIC FUNCTIONS IN COMMUTATIVE ALGEBRAS

Let $\mathbb{A}$ be an arbitrary n-dimensional $(1 \leq n < \infty)$ commutative associative algebra with unit over the field of complex number $\mathbb{C}$. E. Cartan proved that in $\mathbb{A}$ there exist a basis $\{I_k\}_{k=1}^n$ such that the first m basis vectors $I_1, I_2, \ldots, I_m$ are idempotents and another vectors $I_{m+1}, I_{m+2}, \ldots, I_n$ are nilpotents. The element $1 = I_1 + I_2 + \cdots + I_m$ is the unit of $\mathbb{A}$.

In the algebra $\mathbb{A}$ we consider the vectors $e_1, e_2, \ldots, e_d$, $2 \leq d \leq 2n$. Let these vectors have the following decomposition in the basis of the algebra:

$$e_j = \sum_{r=1}^{n} a_{jr}\, I_r\,, \quad a_{jr} \in \mathbb{C}, \quad j = 1, 2, \ldots, d. \tag{1}$$

Throughout this paper, we will assume that at least one of the vectors $e_1, e_2, \ldots, e_d$ is invertible. This condition ensures the uniqueness of the σ-derivative.

For the element $\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d$, where $x_1, x_2, \ldots, x_d \in \mathbb{R}$, the complex numbers

$$\xi_u := x_1a_{1u} + x_2a_{2u} + \cdots + x_da_{du}, \qquad u = 1, 2, \ldots, m$$

forms the spectrum of the point ζ.

Consider in the algebra $\mathbb{A}$ a linear span

$$E_d := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d : \ x_1, x_2, \ldots, x_d \in \mathbb{R}\}$$

generated by the vectors $e_1, e_2, \ldots, e_d$ of $\mathbb{A}$.

Next, the assumption is essential: for each fixed $u \in \{1, 2, \ldots, m\}$ at least one of the numbers $a_{1u}, a_{2u}, \ldots, a_{du}$ belongs to $\mathbb{C} \setminus \mathbb{R}$.

We identify a domain Ω in the space $\mathbb{R}^d$ with the domain

$$\Omega := \{\zeta = x_1e_1 + x_2e_2 + \cdots + x_de_d : (x_1, x_2, \ldots, x_d) \in S\} \text{ in } E_d \subset \mathbb{A}.$$

Definition 1 [1]. *We will call the continuous function $\Phi : \Omega \to \mathbb{A}$ monogenic in the domain $\Omega \subset E_d$ if Φ is differentiable in the sense of Gâteaux at every point of this domain, that is, if for each $\zeta \in \Omega$ there exists an element $\Phi'(\zeta)$ of the algebra $\mathbb{A}$ such that the equality*

$$\lim_{\varepsilon \to 0+0} \frac{\Phi(\zeta + \varepsilon h) - \Phi(\zeta)}{\varepsilon} = h\Phi'(\zeta) \quad \forall\, h \in E_d \tag{2}$$

holds. $\Phi'(\zeta)$ is called the Gâteaux derivative of the function Φ at the point ζ.

The theory of monogenic functions in commutative algebras is well developed in the works of the author and his colleagues S. A. Plaksa, S. V. Gryshchuk and R. P. Pukhtaievych. Monogenic functions are some analog of analytic functions in commutative algebras.

At the end of book [2], V. Kravchenko poses 5 open problems. In the fourth problem, Kravchenko points out the need to construct a pseudoanalytic function theory in multidimensional case.

Our work is an attempt to solve the Kravchenko problem in the case of any finite-dimensional commutative associative algebra. Namely, by developing the ideas of L. Bers and G. Polozhii, σ-monogenic functions will be introduce in any commutative associative algebra.

Let a function $\Phi : \Omega \to \mathbb{A}$ be of the form

$$\Phi(\zeta) = \sum_{k=1}^{n} U_k(x_1, x_2, \ldots, x_d)\, I_k\,. \tag{3}$$

Let σ be a collection of n $\mathbb{A}$-valued functions:

$$\sigma = (\sigma_1, \sigma_2, \ldots, \sigma_n),$$

where $\sigma_k = \sigma_k(x_1, x_2, \ldots, x_d) = \sigma_k(\zeta)$, $k = 1, 2, \ldots, n$, is a function in $\mathbb{A}$.

For a vector $h = \sum\limits_{j=1}^{d} h_je_j$, $h_j \in \mathbb{R}$, we denote

$$\Delta_{\varepsilon,h,\sigma}\Phi(\zeta) := \sum_{k=1}^{n} \sigma_k(\zeta)\Big(U_k(x_1 + \varepsilon h_1, \ldots, x_d + \varepsilon h_d) - U_k(x_1, \ldots, x_d)\Big)\, I_k\,.$$

Definition 2 *We will call the continuous function $\Phi : \Omega \to \mathbb{A}$ σ-monogenic in the domain $\Omega \subset E_d$ if for each $\zeta \in \Omega$ there exists an element $\Phi'_\sigma(\zeta)$ of the algebra $\mathbb{A}$ such that for every $h \in E_d$ the equality*

$$\lim_{\varepsilon\to 0+0} \frac{\Delta_{\varepsilon,h,\sigma}\Phi(\zeta)}{\varepsilon} = h\,\Phi'_\sigma(\zeta) \qquad (4)$$

holds. $\Phi'_\sigma(\zeta)$ is called σ-derivative of the function Φ at the point ζ.

Remark 1. If for all $k = 1, 2, \ldots, n$ $\sigma_k \equiv 1$, then definition 2 coincides with definition 1, i. e. 1-monogenic function is monogenic.

Remark 2. If $\mathbb{A} \equiv \mathbb{C}$ and for special choice of σ_1, σ_2 the definition (4) coincides with the definition of pseudoanalytic function in the sense of Bers.

Necessary and sufficient conditions for σ-monogeneity have been established. In some low-dimensional algebras, with a special choice of σ, the representation of σ-monogenic functions is obtained using holomorphic functions of a complex variable. We proposed the application of σ-monogenic functions with values in two-dimensional biharmonic algebra to representation of solutions of two-dimensional biharmonic equation. The announced results are published in the paper [3].

R E F E R E N C E S

1. Mel'nichenko I. P. *The representation of harmonic mappings by monogenic functions*, Ukr. Math. J., Vol. 27, no. 5, 499–505 (1975).

2. Kravchenko V. V. *Applied pseudoanalytic function theory*, Birkhäuser, Series: Frontiers in Mathematics, Basel, 2008.

3. Shpakivskyi V. S. *σ-monogenic functions in commutative algebras*, Proceedings of the International Geometry Center, Vol. 16, no. 1, 17–41 (2023).

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AN EXTREMAL PROBLEM FOR A MOSAIC SYSTEM OF POINTS

In the geometric theory of functions of a complex variable, the well-known direction is related to the estimates of the products of the inner radii of pairwise nonoverlapping domains. This direction is called extreme problems in classes of pairwise nonoverlapping domains [1]. One of the problems of this type is considered in the present work.

Let the numbers $n, m, d \in \mathbb{N}$ be fixed.

The system of points $A_{n,m} = \{a_{k,p} \in \mathbb{C} : k = \overline{1,n}, p = \overline{1,m}\}$, is called an (n, m)-ray system of points, if, for all $k = \overline{1,n}$, the following relations hold:

$$0 < |a_{k,1}| < \ldots < |a_{k,m}| < \infty;$$

$$\arg a_{k,1} = \arg a_{k,2} = \ldots = \arg a_{k,m} =: \theta_k;$$
$$0 = \theta_1 < \theta_2 < \ldots < \theta_n < \theta_{n+1} := 2\pi.$$

For such systems of points, let us consider the following quantities:

$$\alpha_k = \frac{1}{\pi}\left[\theta_{k+1} - \theta_k\right],\ k = \overline{1,n},\ \alpha_{n+1} := \alpha_1,\ \alpha_0 := \alpha_n, \sum_{k=1}^{n} \alpha_k = 2.$$

For any (n, m)-equiangular ray system of points $A_{n,m} = \{a_{k,p}\}$, we consider the following controlling functional

$$M\left(A_{n,m}\right) = \prod_{k=1}^{n}\prod_{p=1}^{m}\left[\chi\left(|a_{k,p}|^{\frac{1}{\alpha_k}}\right) \cdot \chi\left(|a_{k,p}|^{\frac{1}{\alpha_{k-1}}}\right)\right]^{\frac{1}{2}} \cdot |a_{k,p}|,$$

where $\chi(t) = \frac{1}{2} \cdot \left(t + t^{-1}\right)$.

Consider the system of angular domains:

$$P_k\left(A_{n,m}\right) = \{w \in \mathbb{C} : \theta_k < \arg w < \theta_{k+1}\}, \quad k = \overline{1,n}.$$

For a fixed number $\beta, R \in \mathbb{R}^+$, $0 < \beta < \frac{2\pi}{n}$, consider the unique branch of the multibranch analytic function

$$z_k(w) = \frac{i}{R^{\frac{1}{\alpha_k}} \cdot \sin\frac{\beta}{\alpha_k}} \cdot \left(-\left(e^{-i\theta_k} w\right)^{\frac{1}{\alpha_k}} + R^{\frac{1}{\alpha_k}} \cdot \cos\frac{\beta}{\alpha_k}\right). \tag{1}$$

For each $k = \overline{1,n}$, it realizes the one-sheet conformal mapping of the domain P_k onto the right half-plane $\mathrm{Re} z > 0$.

For each $k = \overline{1,n}$ we denote

$$\begin{aligned}
&\Omega_j^{(k)} := \left\{z :\ |z - i\varrho_j| = r_j, 0 \le \arg z \le \tfrac{\pi}{2}, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\}, j = \overline{1,m},\\
&\Omega_j^{(k)} := \left\{z :\ |z - i\varrho_j| = r_j, -\tfrac{\pi}{2} \le \arg z \le 0, \varrho_j \in \mathbb{R}, r_j \in \mathbb{R}^+\right\},\\
&j = \overline{m+1, 2m},
\end{aligned} \tag{2}$$

where

$$\varrho_1 + r_1 > \varrho_2 + r_2 > \ldots > \varrho_{2m} + r_{2m}.$$

Let $\{b_k\}_{k=1}^{n} \subset \mathbb{C}$ be a set of points such that

$$b_k \in P_k,\ \arg b_k - \theta_k = \beta,\ |b_k| = R,\ k = \overline{1,n}.$$

Let, for each fixed k, $k=\overline{1,n}$, $\left\{L_j^{(k)}\right\}_{j=1}^{2m}$ – be a collection of curves such that

$$\begin{array}{l} L_j^{(k)} \subset \overline{P_k},\ b_k \in L_j^{(k)},\ j=\overline{1,2m}, \\ a_{k,p} \in L_{m-p+1}^{(k-1)},\ a_{k,p} \in L_{m+p}^{(k)},\ p=\overline{1,m}, \\ z_k:\ L_j^{(k)} \to \Omega_j^{(k)},\ j=\overline{1,2m}. \end{array} \tag{3}$$

It is easy to see from relations (1), (2), (3) that

$$z_k\left(b_k\right)=1,\ z_k\left(a_{k+1,p}\right)=i\lambda_p,\ z_k\left(a_{k,p}\right)=-i\lambda_{m+p},$$

$$a_{n+1,p}:=a_{1,p},\ \lambda_t>0, t=\overline{1,2m}, k=\overline{1,n}, p=\overline{1,m}.$$

For each $k=\overline{1,n}$ we denote the corresponding systems of points by

$$D_{2m,d}^{(k)}=\left\{c_{j,s}^{(k)} \in L_j^{(k)}:\ 0<\left|\arg z_k\left(c_{j,1}^{(k)}\right)\right|<\left|\arg z_k\left(c_{j,2}^{(k)}\right)\right|< \right.$$

$$\left. <\ldots<\left|\arg z_k\left(c_{j,d}^{(k)}\right)\right|<\frac{\pi}{2},\ j=\overline{1,2m},\ s=\overline{1,d}\right\}.$$

The system of points

$$AD_{n,m,d}=\bigcup_{k=1}^{n} D_{2m,d}^{(k)} \bigcup A_{n,m}$$

will be called mosaic.

For any mosaic system of points $AD_{n,m,d}$, we consider the following "controlling" functional

$$\mu\left(AD_{n,m,d}\right):=\prod_{k=1}^{n}\left(\left|a_{k,p}\right|\cdot\left|c_{j,s}^{(k)}\right|^{2d}\right)^{1-\frac{1}{\alpha_k}}.$$

For the mosaic point system, the valid result obtained is [2].

R E F E R E N C E S

1. A.K. Bakhtin, G.P. Bakhtina, Yu.B. Zelinskii *Topological-algebraic structures and geometric methods in complex analysis*, Inst. Math. NAS Ukraine, Kiev, (2008).

2. Targonskii A., Bondar S. *An extremal problem for a mosaic system of points in the case of an additional set of points on a circle*, Journal of Mathematical Sciences. **284**(1), 400–409 (2024).

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ANALOG OF HILLE'S THEOREM FOR FINITE-DIMENSIONAL COMMUTATIVE ALGEBRA

We prove that a locally bounded and differentiable in the sense of Gâteaux function given in a finite-dimensional commutative Banach algebra over the complex field is also differentiable in the sense of Lorch. This holds for a functions defined in any domain of any real subspace which satisfies one condition: images of this subspace under all characters of the algebra are whole complex plane.

R E F E R E N C E S

1. Plaksa S. A. Commutative algebras associated with classic equations of mathematical physics, Advances in Applied Analysis, Trends in Mathematics. — Basel: Springer, 2012. — P. 177 – 223.

2. Plaksa S. A. and Shpakovskii V. S. *Constructive description of monogenic functions in a harmonic algebra of the third rank*, Ukr. Math. J., **62** (2011), no. 8, 1251–1266.

3. Plaksa S. A. *On differentiable and monogenic functions in a harmonic algebra*, Zb. Pr. Inst. Mat. NAN Ukr. **14** (1), 210–221 (2017)

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THE ESTIMATES OF THE INNER RADII OF SYMMETRIC NON-OVERLAPPING DOMAINS

Let $\mathbb{N}$ and $\mathbb{R}$ be the sets of natural and real numbers, respectively, $\mathbb{C}$ be the complex plane, $\overline{\mathbb{C}} = \mathbb{C}\bigcup\{\infty\}$ be the Riemann sphere, and $r(B,a)$ be the inner radius of the domain $B \subset \overline{\mathbb{C}}$ with respect to the point $a \in B$.

Consider the different non-overlapping domains $B_0, B_1, ..., B_n$ ($B_p \cap B_j = \emptyset$ for $p \neq j$, $p, j = \overline{0, n}$) such that $a_0 = 0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \mathbb{C}$, $k = \overline{1, n}$, moreover domains $B_1, ..., B_n$ have symmetry with respect to unit circle, and for $\gamma \in (0, n]$ consider the value

$$I_n(\gamma) = r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k). \quad (1)$$

Problem. (see [1]) *For any fixed* $\gamma \in (0, n]$ *to find the maximum of the functional (1) and to show that this maximum is reached for some configuration of the domains* B_k *and points* a_k, $k = \overline{0, n}$, *which has n-fold symmetry.*

This problem is one of the problems of the geometric function theory. The problem has a solution only if $\gamma \leqslant n$ as soon as $\gamma = n + \epsilon, \epsilon > 0$, the problem has no solution. Currently it still unsolved in general, only partial results are known (see, f.e. [2]).

The following theorem holds (prove see in [3]).

Theorem 1. *Let* $n = \overline{4, 7}$, $1 < \gamma \leqslant \gamma_n$, $\gamma_4 = 1,6$, $\gamma_5 = 1,65$, $\gamma_6 = 1,7$, $\gamma_7 = 1,77$. *Then for any different system of points* $\{a_k\}_{k=0}^{n}$ *such that* $a_0 = 0$, $|a_k| = 1$, $k = \overline{1, n}$ *and for any different system of non-overlapping domains* $\{B_k\}_{k=0}^{n}$ *such that* $a_0 \in B_0 \subset \overline{\mathbb{C}}$, $a_k \in B_k \subset \mathbb{C}$, $k = \overline{1, n}$, *moreover domains* $\{B_k\}_{k=1}^{n}$ *have symmetry with respect to unit circle, the following inequality holds*

$$r^{\gamma}(B_0, 0) \prod_{k=1}^{n} r(B_k, a_k) \leqslant \left(\frac{4}{n}\right)^{n} \frac{\left(\frac{2\gamma}{n^2}\right)^{\frac{\gamma}{n}}}{\left(1 - \frac{2\gamma}{n^2}\right)^{\frac{n}{2} + \frac{\gamma}{n}}} \left(\frac{n - \sqrt{2\gamma}}{n + \sqrt{2\gamma}}\right)^{\sqrt{2\gamma}}.$$

Equality is attained if a_k *and* B_k, $k = \overline{0, n}$ *are, respectively, poles and circular domains of the quadratic differential*

$$Q(w)dw^2 = -\frac{\gamma w^{2n} + 2(n^2 - \gamma)w^n + \gamma}{w^2(w^n - 1)^2} dw^2.$$

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REFERENCES

1. Dubinin, V.N. *Symmetrization in the geometric theory of functions of a complex variable.* Russ. Math. Surv., **49**(1), 1-79 (1994).

2. Bakthin, A.K., Bakhtina, G.P., Zelinskii, Yu.B. *Topological-algebraic structures and geometric methods in complex analysis.* In Proceedings of the Institute of Mathematics of NAS of Ukraine. — Kyiv: Institute of Mathematics of NAS of Ukraine, (2008).

3. Bakthin, A.K., Vyhivska, L.V. *The problem of extremal decomposition of a complex plane with free poles on a circle.* Journal of Mathematical Sciences, **260**(5), 630-650 (2022).

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ASYMPTOTIC BEHAVIOR OF THE LOGARITHMIC DERIVATIVE OF THE BLASCHKE PRODUCT WITH SLOWLY INCREASING COUNTING FUNCTION OF ZEROS

Let $a_n = 1-r_n e^{i\theta_n}$ $(-\pi/2 < \theta_n < \pi/2)$ be a sequence of zeros of the Blaschke product $B(z) = \prod\limits_{n=1}^{+\infty} \dfrac{\overline{a}_n}{|a_n|} \cdot \dfrac{a_n - z}{1-\overline{a}_n z}$, $n(t) = n(t, B)$ be a number of (a_n) on the disc $\{z\colon |z| \le t\}$ such that $1 - r_n \le t$, $0 < t < 1$, $r_n \to 0+$ as $n \to +\infty$. L denotes class of slowly increasing at a point 1 functions v, that is $v(t)$ be continuous on $[0, 1)$, $v(0) = 0$, $v(t) \sim v\left((1 + t)/2\right)$, $t \to 1-$.

We denote by $\mathcal{B}(v)$, $v \in L$, a set of Blaschke products B which zeros satisfy the condition $\varlimsup\limits_{t\to 1-} n(t, B)/v(t) < +\infty$.

Let $\Gamma_m = \bigcup\limits_{j=1}^{m} \{z\colon |z| < 1, \arg(1 - z) = -\psi_j\} = \bigcup\limits_{j=1}^{m} l_{\psi_j}$, $-\pi/2 + \eta < \psi_1 < \psi_2 < \ldots < \psi_m < \pi/2 - \eta$, $0 < \eta < 1$, be a finite system of rays; $\mathcal{B}(v; \Gamma_m)$ be a subclass of products B of the class $\mathcal{B}(v)$ with zeros (a_n) on Γ_m; $n(t, \psi_j) = n(t, \psi_j; B)$ be the number of zeros of the product B on the ray l_{ψ_j} such that $1 - r_n \le t$.

For $\widetilde{v} \in L$ we set $v(t) = \int\limits_0^t \dfrac{\widetilde{v}(x)}{1-x} dx$, $\quad 0 \le t < 1$, $\quad v(0) = 0$. It is easy to see that $v \in L$ and $\widetilde{v}(t) = o\left(v(t)\right)$, $t \to 1-$.

Theorem 1. *Let $\widetilde{v} \in L$, $\Delta_j \ge 0$, $B \in \mathcal{B}(v; \Gamma_m)$ and for each $j = \overline{1, m}$ $n(t; \psi_j) = \Delta_j v(t) + o\left(\widetilde{v}(t)\right)$, $t \to 1-$. Then for $z = 1 - re^{-i\varphi}$, $\varphi \in \left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right) \setminus \left(\bigcup\limits_{j=1}^{m} \theta_{\psi_j}\right)$,*

$$(1-z)\frac{B'(z)}{B(z)} = -i\sum_{j=1}^{m} \Delta_j \left(2\psi_j + \pi\, sign(\varphi - \psi_j)\right) \widetilde{v}(1-r) + o\left(\widetilde{v}(1-r)\right), \ r \to 0+, \tag{1}$$

moreover for any $\delta > 0$ relation (1) *holds uniformly relative to φ on the set $\left(-\dfrac{\pi}{2}; \dfrac{\pi}{2}\right) \setminus \left(\bigcup\limits_{j=1}^{m} (\psi_j - \delta, \psi_j + \delta)\right)$.*

Theorem 2. *Let $\widetilde{v} \in L$, $\Delta_j \ge 0$, $B \in \mathcal{B}(v; \Gamma_m)$ and relation* (1) *holds. Then $n(t; \psi_j) = (1 + o(1))\, \Delta_j v(t)$, $\quad t \to 1-$.*

REFERENCES

1. Zabolotskyi, M.V., Gal, Y.M., Mostova, M.R. *Logarithmic derivative of the Blaschke product with slowly increasing counting function of zeros*, Bukovinian Math. Journal. **9**(1), 164–170 (2021).

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ASYMPTOTIC BEHAVIOR OF SUBHARMONIC IN $\mathbb{R}^m$, $m \geq 3$, FUNCTIONS OF SLOW GROWTH

By $SH_m(0)$ we denote a class of functions of order zero subharmonic in $\mathbb{R}^m$, $m \geq 3$. Further, by $SH_m^-(0)$ we denote a subclass of functions u from $SH_m(0)$ such that u are harmonic functions beyond the negative semiaxis Ox_1.

A nonnegative nondecreasing unbounded function on $[0; +\infty)$ is called a comparison function. By L we denote the set of continuously differentiable comparison functions v such that $tv'(t)/v(t) \to 0$ as $t \to +\infty$. For $v \in L$ we set $v_1(r) = \int\limits_1^r v(t)/t dt$.

We denote
$$A(m) = \sum_{k=1}^{m-2} C_{m-1}^k I_{m-1}(m-2-k),$$
where
$$I_n(k) = \int\limits_1^{+\infty} \frac{t^k dt}{(t+1)^n},\ n \in \mathbb{N},\ n \geq 2,\ k = 0, 1, ..., n-2.$$

Let μ is Riesz measure of a function u, $n(t,u) = \mu\big(\{x : |x| \leq t\}\big)$, $N(t,u) = (m-2)\int\limits_1^t n(\tau,u)/\tau^{m-1} d\tau$.

Theorem 1. *Suppose that* $u \in SH_m^-(0)$, $v \in L$, $u(r) = u(r, 0, ..., 0)$. *(A) If*
$$N(t,u) = v_1(t) + o(v(t)),\ t \to +\infty,$$
then
$$u(r) = v_1(r) + A(m)v(r) + o(v(r)),\ r \to +\infty. \tag{1}$$
(B) Conversely, if (1) is true, then
$$N(t,u) = (1 + o(1))v_1(t),\ t \to +\infty.$$

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THE NORMAL DISTRIBUTION LAW OF RANDOM QUATERNIONS AND ITS APPLICATION

In recent years, research on quaternion normal distribution has been devoted to works in which authors study various approaches to developing and describing this distribution using complex mathematical methods [1-3]. This paper presents a visual model of the normal distribution of random quaternions (under specified conditions), which is a generalization of the three-sigma rule of the classical normal distribution to the case of standard deviations of quaternions.

Let us have a finite sample from a set of random quaternions Q:

$$Q_m = a + bi + cj + dk, \quad m = 1, 2, \ldots, n,$$

whose components $a, b, c, d \in \mathbb{R}$ are normally distributed random variables with corresponding probabilities $p_1, p_2, p_3, p_4 \in \mathbb{R}$. Then we write the probabilities of quaternions appearing in the sample as quaternion probabilities:

$$P_m = p_1 + p_2 i + p_3 j + p_4 k, \quad p_1, p_2, p_3, p_4 \in \mathbb{R}, \quad m = 1, 2, \ldots, n.$$

The *mathematical expectation* of random quaternions Q has two formulas:

$$M_R(Q) = \sum_{m=1}^{n} Q_m P_m; \quad M_L(Q) = \sum_{m=1}^{n} P_m Q_m. \tag{1}$$

Using the right-hand and left-hand mathematical expectations (1), we obtain four formulas for the variances of random quaternions Q:

$$D_{RR}(Q) = \sum_{m=1}^{n} \left(Q_m - M_R(Q)\right)^2 P_m; D_{RL}(Q) = \sum_{m=1}^{n} \left(Q_m - M_L(Q)\right)^2 P_m;$$
$$D_{LL}(Q) = \sum_{m=1}^{n} P_m \left(Q_m - M_L(Q)\right)^2; D_{LR}(Q) = \sum_{m=1}^{n} P_m \left(Q_m - M_R(Q)\right)^2. \tag{2}$$

According to (2), we have four standard deviations:

$$\sigma_{RR}(Q) = \sqrt{D_{RR}(Q)}; \quad \sigma_{RL}(Q) = \sqrt{D_{RL}(Q)};$$
$$\sigma_{LL}(Q) = \sqrt{D_{LL}(Q)}; \quad \sigma_{LR}(Q) = \sqrt{D_{LR}(Q)}. \tag{3}$$

Let us consider a sample of unique random quaternions in the sense that each quaternion will occur only once in this sample, i.e., the probabilities of encountering all quaternions are equal. In this case, we will write the quaternion-valued probabilities as:

$$p_m = p(1 + i + j + k), \quad m = 1, 2, \ldots, n, \quad p \in \left[0; \frac{1}{2}\right], \quad p = \frac{1}{2n}. \qquad (4)$$

Writing down formulas (1) – (3) and taking into account (4), we construct a visualization of the normal distribution of random quaternions (Fig. 1). The model is implemented in the Python programming language.

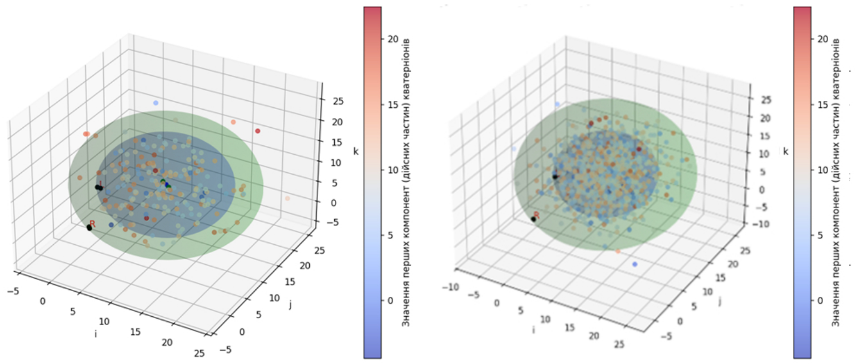

Рис. 1: Normal distribution of 200 and 1000 quaternions at $M = 10, \sigma = 5$.

Empirically, we constructed distributions of random quaternions with volumes of 100, 150, 200, 500, and 1000 points at $M = 10, \sigma = 5$. Based on the constructed models, we concluded that the visual model of the normal distribution of random quaternions looks like two spheres, with the left sphere usually closer to the center of dispersion and the right sphere further from the center of dispersion. Fig.1 shows that the larger the sample size, the greater the number of quaternions located closer to the center of dispersion. In this regard, we formulate the rule of standard deviations of quaternions as follows: almost 68% of quaternion points fall within the sphere with the smaller radius (i.e., the left standard deviations play the role of 1σ), and almost 99,7% of quaternions fall within the sphere with the larger radius (i.e., the right standard deviations play the role of 3σ). Elements located outside the larger sphere are at distances greater than the radii of standard deviations from the center of dispersion and therefore fall outside the 3σ limits.

We can see that the model (Fig.1) has similarities with real physical objects – spherical star clusters studied in astrophysics. This makes it possible to apply the obtained model, for example, to simulate the interaction of stars in clusters or as initial conditions for the evolution of star motion over time.

R E F E R E N C E S

1. Suzuki T. *Quaternion gaussian integral and its application to a geometry of quaternion gaussian distributions.* International Conference on Mathematics: Pure, Applied and Computation, 1-7 (2019).

2. Chelidze G., Vakhania N. *On a quaternion valued Gaussian random variables.* Stud. Univ. Babeş-Bolyai Math. **56** (2), 295—298 (2011).

3. Loots M. T. *The development of the quaternion normal distribution*: Master's dissertation, University of Pretoria, Department of Statistics, Pretoria, South Africa (2010).

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ON THE FUNCTIONS RELATED WITH THE ANALYTIC SOLUTIONS OF THE CAUCHY PROBLEM FOR WAVE AND HEAT EQUATIONS

Notations, definitions and more references on the topic are in [1,2]. We investigate properties of entire solutions of the Cauchy problem for one-dimensional homogeneous hyperbolic equation. Considering analytic continuation of the solutions given by the D'Alambert formula we have found some conditions providing L-index boundedness in the direction for some functions related with the solutions. In particular, for homogomogeneous wave equation $c^2\frac{\partial^2}{\partial x^2}u(x,t) = \frac{\partial^2}{\partial t^2}u(x,t)$ with initial conditions $u(x,0) = \varphi(x)$, $u_t(x,0) = \psi(x)$ its solution has the form

$$u(x,t) = \tfrac{\varphi(x+ct)+\varphi(x-ct)}{2} + \tfrac{1}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha.$$

We study the function $H(x,t) = \frac{\varphi(x+ct)+\varphi(x-ct)}{2} + \frac{e}{2c}\int\limits_{x-ct}^{x+ct}\psi(\alpha)d\alpha$, where x, t, $c \in \mathbb{C}$, e is a positive constant which is determined with some conditions by the functions φ and ψ. Our main result gives sufficient conditions of boundedness of L-index in a direction $\mathbf{b}$ for the functions H. Its proof uses known sufficent conditions for the sum of entire functions. At the end, we pose open problems concerning conditions of the directional L-index boundedness for analytic solutions of the Cauchy problem of the heat equation.

An entire function $F(z)$, $z \in \mathbb{C}^n$, is called *a function of bounded L-index in a direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$, if there exists $m_0 \in \mathbb{Z}_+$ such that for every $m \in \mathbb{Z}_+$ and every $z \in \mathbb{C}^n$

$$\frac{1}{m!L^m(z)}\left|\frac{\partial^m F(z)}{\partial \mathbf{b}^m}\right| \le \max\left\{\frac{1}{k!L^k(z)}\left|\frac{\partial^k F(z)}{\partial \mathbf{b}^k}\right| : 0 \le k \le m_0\right\},$$

where $\frac{\partial^0 F(z)}{\partial \mathbf{b}^0} := F(z)$, $\frac{\partial F(z)}{\partial \mathbf{b}} := \sum_{j=1}^{n} \frac{\partial F(z)}{\partial z_j} b_j = \langle \mathbf{grad} F, \overline{\mathbf{b}} \rangle$, $\frac{\partial^k F(z)}{\partial \mathbf{b}^k} := \frac{\partial}{\partial \mathbf{b}}\left(\frac{\partial^{k-1} F(z)}{\partial \mathbf{b}^{k-1}}\right)$, $k \geq 2$.

The least such integer $m_0 = m_0(\mathbf{b})$ is called the *L-index in the direction* $\mathbf{b} \in \mathbb{C}^n \setminus \{\mathbf{0}\}$ *of the entire function* $F(z)$ *and is denoted by* $N_{\mathbf{b}}(F, L) = m_0$. If such m_0 does not exist then F is called *a function of unbounded L-index in the direction* $\mathbf{b}$ and we suppose that $N_{\mathbf{b}}(F, L) = \infty$. If $L(z) \equiv 1$ then $F(z)$ is called *a function of bounded index in the direction* $\mathbf{b}$ and $N_{\mathbf{b}}(F) = N_{\mathbf{b}}(F, 1)$.

Theorem 1. *Let $\varphi : \mathbb{C} \to \mathbb{C}$, $\psi : \mathbb{C} \to \mathbb{C}$ be entire functions, $l : \mathbb{C} \to \mathbb{R}_+$, $l \in Q$, $c \in \mathbb{C} \setminus \{0\}$. The following conditions are satisfied:*

1. *the function $F(x, t) = \varphi(x + ct) + \varphi(x - ct)$ is of bounded L-index in the direction $\mathbf{b} = (b_1, b_2)$, where the function L is given by the formula $L(x, t) = \max\{l(x - ct), l(x + ct)\}$ and $b_1 - b_2 c \neq 0$;*
2. *For every point $t \in \mathbb{C}$ the function $\Psi_t(\tau) = \int\limits_{(b_1 - b_2 c)\tau}^{2ct + (b_1 + b_2 c)\tau} \psi(\alpha) d\alpha \not\equiv 0$;*
3. *For every point $t \in \mathbb{C}$ one has $\max\limits_{|t' - \tau_0| = 2|t - \tau_0|} \Big| \int\limits_{(b_1 - b_2 c)t'}^{2ct + (b_1 + b_2 c)t'} \psi(s) ds \Big| \leq$*
$$\leq c \max_{0 \leq k \leq N_{\mathbf{b}}(F_\alpha, L_\alpha)} \big\{ |(b_1 - cb_2)^k \cdot \varphi^{(k)}((b_1 - b_2 c)t) +$$
$$+ (b_1 + cb_2)^k \cdot \varphi^{(k)}((b_1 + (b_2 + 2)c)t)| / (k! L^k((c_1 + b_1)t, (1 + b_2)t)) \big\}.$$
where τ_0 is chosen such that $\Psi_t(\tau_0) \neq 0$;
4. *There exist such positive constant e such that*
$$\max\left\{ \Big| \int\limits_{(b_1 - b_2 c)t'}^{2ct + (b_1 + b_2 c)t'} \psi(s) ds \Big| : |t' - \tau_0| = 2\lambda_2^{\mathbf{b}}(1) / L(c_1 t + b_1 \tau_0, t + b_2 \tau_0) \right\} \leq$$
$$\leq e \Bigg| \int\limits_{(b_1 - b_2 c)\tau_0}^{2ct + (b_1 + b_2 c)\tau_0} \psi(\alpha) d\alpha \Bigg|. \qquad (1)$$

If $|\varepsilon| < \frac{1}{2e}$ then the function $H(x, t) = \frac{\varphi(x + ct) + \varphi(x - ct)}{2} + \frac{\varepsilon}{2c} \int\limits_{x - ct}^{x + ct} \psi(\alpha) d\alpha$ is of bounded L-index in the direction $\mathbf{b}$ with $N_{\mathbf{b}}(H, L) \leq N_{\mathbf{b}}(F, L)$.

R E F E R E N C E S

1. Bandura, A. *On the functions related with the analytic solutions of the Cauchy problem for wave and heat equation*, Precarpathian bulletin of the Schevchenko Scientific Society. Number. **19(73)**, 9–17 (2024). https://doi.org/10.31471/2304-7399-2024-19(73)-9-17

2. Bandura, A., Skaskiv, O., Hural, I., Shehda, L., Smolovyk, L. *Analytic functions in a unit polydisc: conditions of directional boundedness of L-index for the sum of functions with directionally bounded L-index,* Journal of Mathematical Sciences. **288**(2), 155–162 (2025). https://doi.org/10.1007/s10958-025-07674-8

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ON CONVERGENCE OF SOME CLASS OF MAPPINGS IN METRIC SPACES

Everywhere further, (X, d, μ) and (X', d', μ') are metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Let G and G' be domains with finite Hausdorff dimensions α and $\alpha' \geqslant 2$ in (X, d, μ) and (X', d', μ'), respectively. For $x_0 \in X$ and $r > 0$, $S(x_0, r)$ denotes the sphere $\{x \in X : d(x, x_0) = r\}$. Put $d(E) := \sup\limits_{x,y\in E} d(x, y)$. Given $0 < r_1 < r_2 < \infty$, denote $A = A(x_0, r_1, r_2) = \{x \in X : r_1 < d(x, x_0) < r_2\}$. Let $p \geqslant 1$ and $q \geqslant 1$, and let $Q : G \to [0, \infty]$ be a measurable function. Due to [1], a homeomorphism $f : G \to G'$ is called a *ring Q-homeomorphism at a point $x_0 \in \overline{G}$ with respect to (p, q)-moduli*, if the inequality

$$M_p(f(\Gamma(S(x_0, r_1), S(x_0, r_2), A(x_0, r_1, r_2))))$$

$$\leqslant \int\limits_{A(x_0,r_1,r_2)\cap G} Q(x) \cdot \eta^q(d(x, x_0))\, d\mu(x)$$

holds for all $0 < r_1 < r_2 < r_0 := d(G)$ and each measurable function $\eta : (r_1, r_2) \to [0, \infty]$ with $\int\limits_{r_1}^{r_2} \eta(r)\, dr \geqslant 1$. We say that $f : G \to G'$ is a *ring Q-homeomorphism at a point $x_0 \in \overline{G}$, if the latter is true for $p = \alpha'$ and $q = \alpha$.*

We say that the condition of the *complete divergence of paths* is satisfied in $D \subset X$ if for any different points y_1 and $y_2 \in D$ there are some w_1,

$w_2 \in \partial D$ and paths $\alpha_2 : (-2, -1] \to D$, $\alpha_1 : [1, 2) \to D$ such that 1) α_1 and α_2 are subpaths of some geodesic path $\alpha : [-2, 2] \to X$, that is, $\alpha_2 := \alpha|_{(-2,-1]}$ and $\alpha_1 := \alpha|_{[1,2)}$; 2) the geodesic path α joins the points w_2, y_2, y_1 and w_1 such that $\alpha(-2) = w_2$, $\alpha(-1) = y_2$, $\alpha(1) = y_1$, $\alpha(2) = w_1$.

Let $\dim X = n$. For each real number $n \geqslant 1$, we define *the Loewner function* $\Phi_n : (0, \infty) \to [0, \infty)$ on X as $\Phi_n(t) = \inf\{M_n(\Gamma(E, F, X)) : \Delta(E, F) \leqslant t\}$, where the infimum is taken over all disjoint nondegenerate continua E and F in X and

$$\Delta(E, F) := \frac{\mathrm{dist}\,(E, F)}{\min\{d(E), d(F)\}}\,.$$

A pathwise connected metric measure space (X, μ) is said to be a *n-Loewner space,* if the Loewner function $\Phi_n(t)$ is positive for all $t > 0$. Observe that, $\mathbb{R}^n$ and $\mathbb{B}^n \subset \mathbb{R}^n$ are Loewner spaces (see Theorem 8.2 and Example 8.24(a) in [2]). A domain D in $\mathbb{R}^n$ is called *QED-domain* if

$$M(\Gamma(E, F, \mathbb{R}^n)) \leqslant A \cdot M(\Gamma(E, F, D))$$

for some finite number $A \geqslant 1$ and all continua E and F in D. Given a domain D in X, a measurable function $Q : D \to [0, \infty]$, a compact set $K \subset D$ and numbers $A, \delta > 0$ denote by $\mathfrak{F}^{A,\delta}_{K,Q}(D)$ a family of all ring Q-homeomorphisms $f : D \to X'$ such that $D_f := f(D)$ is a compact QED-subdomain of X' with general A in the definition of a QED-domain and, in addition, $d'(f(K), \partial D_f) \geqslant \delta$.

Theorem. *Let (X, d, μ) and (X', d', μ') be metric spaces with metrics d and d' and locally finite Borel measures μ and μ', correspondingly. Let D be a domain in X in which the condition of the complete divergence of paths is satisfied. and let $f_n : D \to X'$, $n = 1, 2, \ldots$, be a sequence of homeomorphisms of the class $\mathfrak{F}^{A,\delta}_{K,Q}(D)$ which converges to some mapping $f : D \to X'$ locally uniformly. Let X' be a n-Loewner space in which the relation $\mu(B_R) \leqslant C^* R^n$ holds for some constant $C^* \geqslant 1$, for some exponent $n > 0$ and for all closed balls B_R of radius $R > 0$. Let $K = \overline{G}$ and G is a compact subdomain of D. If $Q \in L^1(D)$, then f is a discrete in G.*

If in addition, X' is locally path connected, then f is open. Besides that, if all balls in X' are connected, and all closed balls in X are compact, then f is a homeomorphism.

R E F E R E N C E S

1. Martio, O., Ryazanov, V., Srebro, U., and Yakubov, E. *Moduli in modern mapping theory,* Springer Science + Business Media, LLC, New York (2009).

2. Heinonen, J. *Lectures on Analysis on metric spaces,* Springer Science+Business Media, New York (2001).

Contents

INTERNATIONAL CONFERENCE "COMPLEX AND HYPERCOMPLEX ANALYSIS AND THEIR APPLICATIONS"

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